A T M E®
College of Engineering
Affiliated to VTU, Belagavi, Approved by AICTE, New Delhi and Recognized by Government of Karnataka,
Programs accredited by NBA, New Delhi – **UG:** CV, ECE, EEE and ME (Validity up to June 2025), **UG:** CSE (Validity up to June 2026)

Mathematics-III for EE Engineering

Course Code: BMATE 301

Semester – III

Department of Mathematics

Academic Year 25-26

Mathematics-III for EE Engineering			
Course Code	**BMATE 301**	CIE Marks	50
Teaching Hours/Week (L:T:P: S)	3:1:0:0	SEE Marks	50
Total Hours of Pedagogy	40	Total Marks	100
Credits	03	Exam Hours	03
Examination type (SEE)	Theory		

Course objectives:

- To acquaint the students with differential equations and their applications in electrical engineering
- To find the association between attributes and the correlation between two variables
- Learn to use Fourier series to represent periodical physical phenomena in engineering analysis and to enable the student to express non periodic functions to periodic function using Fourier series and Fourier transforms.
- To learn the basic ideas of the theory of probability and random signals.

Teaching-Learning Process (General Instructions)

These are sample Strategies; which teachers can use to accelerate the attainment of the various course outcomes.

1. Lecturer method (L) needs not to be only traditional lecture method, but alternative effective teaching methods could be adopted to attain the outcomes.
2. Use of Video/Animation to explain functioning of various concepts.
3. Encourage collaborative (Group Learning) Learning in the class.
4. Ask at least three HOT (Higher order Thinking) questions in the class, which promotes critical thinking.
5. Adopt Problem Based Learning (PBL), which fosters students' Analytical skills, develop design thinking skills such as the ability to design, evaluate, generalize, and analyse information rather than simply recall it.
6. Introduce Topics in manifold representations.
7. Show the different ways to solve the same problem with different circuits/logic and encourage the students to come up with their own creative ways to solve them.
8. Discuss how every concept can be applied to the real world - and when that's possible, it helps improve the students' understanding.

Module-1 :Ordinary Differential Equations of Higher Order (8 hours)

Importance of higher-order ordinary differential equations in Electrical & Electronics Engineering applications.

Higher-order linear ODEs with constant coefficients - Inverse differential operator, problems.Linear differential equations with variable Coefficients-Cauchy's and Legendre's differential equations - Problems.

Applications: Application of linear differential equations to L-C circuit and L-C-R circuit.

Self-Study: Finding the solution by the method of undetermined coefficients and method of variation of parameters.

(RBT Levels: L1, L2 and L3)

Module-2: Curve fitting, Correlation and regressions

Principles of least squares, Curve fitting by the method of least squares in the form $y = a + bx$, $y = a + bx + cx^2$, and $y = ax^b$. Correlation, Co-efficient of correlation, Lines of regression, Angle between regression lines, standard error of estimate, rank correlation

Self-study: Fitting of curves in the form $y = a\,e^{bx}$

Module-3 Fourier series.

Periodic functions, Dirchlet's condition, conditions for a Fourier series expansion, Fourier series of functions with period 2π and with arbitrary period. Half rang Fourier series. Practical harmonic analysis.

Application to variation of periodic current.

Self-study: Typical waveforms, complex form of Fourier series

Module-4 Fourier transforms and Z -transforms

Infinite Fourier transforms: Definition, Fourier sine, and cosine transform. Inverse Fourier transforms Inverse Fourier cosine and sine transforms. Problems.

Z-transforms: Definition, Standard z-transforms, Damping, and shifting rules, Problems. Inverse z-transform and applications to solve difference equations

Self-study: Convolution theorems of Fourier and z-transforms

Module-5 Probability distributions

Review of basic probability theory, Random variables-discrete and continuous Probability distribution function, cumulative distribution function, Mathematical Expectation, mean and variance, Binomial, Poisson,Exponential and Normal distribution (without proofs for mean and SD) – Problems.

Sampling Theory: Introduction to sampling distributions, standard error, Type-I and Type-II errors.Student's t-distribution, Chi-square distribution as a test of goodness of fit.

Self-study: Test of hypothesis for means, single proportions only.

Course outcome (Course Skill Set)

At the end of the course, the student will be able to :

1. Understand that physical systems can be described by differential equations and solve such equations
2. Make use of correlation and regression analysis to fit a suitable mathematical model for statistical data
3. Demonstrate the Fourier series to study the behavior of periodic functions and their applications in system communications, digital signal processing, and field theory.
4. To use Fourier transforms to analyze problems involving continuous-time signals and to apply Z-Transform techniques to solve difference equations
5. Apply discrete and continuous probability distributions in analyzing the probability models arising in the engineering field. Demonstrate the validity of testing the hypothesis.

Assessment Details (both CIE and SEE)

The weightage of Continuous Internal Evaluation (CIE) is 50% and for Semester End Exam (SEE) is 50%. The minimum passing mark for the CIE is 40% of the maximum marks (20 marks out of 50) and for the SEE minimum passing mark is 35% of the maximum marks (18 out of 50 marks). The student is declared as a pass in the course if he/she secures a minimum of 40% (40 marks out of 100) in the sum total of the CIE (Continuous Internal Evaluation) and SEE (Semester End Examination) taken together.

Continuous Internal Evaluation:

- There are 25 marks for the CIE's Assignment component and 25 for the Internal Assessment Test component.
- Each test shall be conducted for 25 marks. The first test will be administered after 40-50% of the coverage of the syllabus, and the second test will be administered after 85-90% of the coverage of the syllabus. The average of the two tests shall be scaled down to 25 marks
- Any two assignment methods mentioned in the 22OB2.4, if an assignment is project-based then only one assignment for the course shall be planned. The schedule for assignments shall be planned properly by the course teacher. The teacher should not conduct two assignments at the end of the semester if two assignments are planned. Each assignment shall be conducted for 25 marks. (If two assignments are conducted then the sum of the two assignments shall be scaled down to 25 marks)
- The final CIE marks of the course out of 50 will be the sum of the scale-down marks of tests and assignment/s marks.

Internal Assessment Test question paper is designed to attain the different levels of Bloom's taxonomy as per the outcome defined for the course.

Semester-End Examination:

Theory SEE will be conducted by University as per the scheduled timetable, with common question papers for the course (**duration 03 hours).**

1. The question paper will have ten questions. Each question is set for 20 marks.
2. There will be 2 questions from each module. Each of the two questions under a module (with a maximum of 3 sub-questions), **should have a mix of topics** under that module.
3. The students have to answer 5 full questions, selecting one full question from each module.
4. Marks scored shall be proportionally reduced to 50 marks

Suggested Learning Resources:

Books (Title of the Book/Name of the author/Name of the publisher/Edition and Year)

Text Books

1. **B. S. Grewal**: "Higher Engineering Mathematics", Khanna Publishers, 44th Ed., 2021.
2. **E. Kreyszig**: "Advanced Engineering Mathematics", John Wiley & Sons, 10th Ed., 2018.

Reference Books

1. **V. Ramana:** "Higher Engineering Mathematics" McGraw-Hill Education, 11th Ed., 2017
2. **Srimanta Pal & Subodh C.Bhunia**: "Engineering Mathematics" Oxford University Press, 3rd Ed., 2016.
3. **N.P Bali and Manish Goyal**: "A Textbook of Engineering Mathematics" Laxmi Publications, 10th Ed., 2022.
4. **C. Ray Wylie, Louis C. Barrett:** "Advanced Engineering Mathematics" McGraw – Hill Book Co., New York, 6th Ed., 2017.

5. **Gupta C.B, Sing S.R and Mukesh Kumar:** "Engineering Mathematic for Semester I and II", Mc-Graw Hill Education(India) Pvt. Ltd 2015.
6. **H.K. Dass and Er. Rajnish Verma:** "Higher Engineering Mathematics" S.Chand Publication, 3rd Ed.,2014.
7. **James Stewart: "**Calculus" Cengage Publications, 7th Ed., 2019.

Web links and Video Lectures (e-Resources):

http://nptel.ac.in/courses.php?disciplineID=111
- http://www.class-central.com/subject/math(MOOCs)
- http://academicearth.org/
- VTU e-Shikshana Program

VTU EDUSAT Program.

Activity Based Learning (Suggested Activities in Class)/ Practical Based Learning

Activity-Based Learning (Suggested Activities in Class)/Practical-Based Learning

- Quizzes
- Assignments
- Seminar

Module-1 – Ordinary Differential Equations of Higher Order

- **Importance of higher-order ordinary differential equations in Electrical & Electronics Engineering applications.**

  Higher-order linear ODEs with constant coefficients - Inverse differential operator, problems.Linear differential equations with variable Coefficients-Cauchy's and Legendre's differential equations - Problems.

- **Applications:** Application of linear differential equations to L-C circuit and L-C-R circuit.

- **Self-Study:** Finding the solution by the method of undetermined coefficients and method of variation of parameters.

Ordinary Differential Equations of higher order

INTRODUCTION:

In this module, we study differential equations of second and higher orders. Differential equations of second order arise very often in physical problems, especially in connection with mechanical vibrations and electric circuits.

A differential equation of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = X \qquad \dots(1)$$

where X is a function of x and $a_1, a_2 \dots, a_n$ are constants is called a linear differential equation of n^{th} order with constant coefficients. Since the highest order of the derivative appearing in (1) is n, it is called a differential equation of n^{th} order and it is called linear.

Using the familiar notation of differential operators:

$$D = \frac{d}{dx}, \; D^2 = \frac{d^2}{dx^2}, \; D^3 = \frac{d^3}{dx^3} \dots, D^n = \frac{d^n}{dx^n}$$

Then (1) can be written in the form

$$\{D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots a_n\} y = X$$

i.e., $$f(D)\, y = X \qquad \dots(2)$$

where $$f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots a_n.$$

Here $f(D)$ is a polynomial of degree n in D

If $x = 0$, the equation

$$f(D)\, y = 0$$

is called a homogeneous equation.

If $x \neq 0$ then the Eqn. (2) is called a non-homogeneous equation.

SOLUTION OF A HOMOGENEOUS SECOND ORDER LINEAR DIFFERENTIAL EQUATION

1. *Solve* $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0.$

Solution. Given equation is $(D^2 - 5D + 6)\,y = 0$

A.E. is $\qquad m^2 - 5m + 6 = 0$

i.e., $\qquad (m-2)\,(m-3) = 0$

i.e., $\qquad m = 2, 3$

$\therefore \qquad m_1 = 2,\ m_2 = 3$

$\therefore$ The roots are real and distinct.

We consider the homogeneous equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

where p and q are constants

$$(D^2 + pD + q)\,y = 0$$

The Auxiliary equations (A.E.) put $D = m$

$$m^2 + pm + q = 0$$

Eqn. (3) is called auxiliary equation (A.E.) or characteristic equation of the D.E. eqn. (quadratic in m, will have two roots in general. There are three cases.

Case (*i*): Roots are real and distinct

The roots are real and distinct, say m_1 and m_2 *i.e.,* $m_1 \neq m_2$

Hence, the general solution of eqn. (1) is

$$y = C_1\, e^{m_1 x} + C_2\, e^{m_2 x}$$

where C_1 and C_2 are arbitrary constant.

Case (*ii*): Roots are equal

The roots are equal *i.e.,* $m_1 = m_2 = m.$

Hence, the general solution of eqn. (1) is

$$y = (C_1 + C_2\, x)\, e^{mx}$$

where C_1 and C_2 are arbitrary constant.

Case (*iii*): Roots are complex

The Roots are complex, say $\alpha \pm i\beta$

Hence, the general solution is

$$y = e^{\alpha x}\,(C_1 \cos \beta\, x + C_2 \sin \beta\, x)$$

where C_1 and C_2 are arbitrary constants.

Note. Complementary Function (C.F.) which itself is the general solution of the D.E.

$\therefore$ The general solution of the equation is

$$y = C_1 e^{2x} + C_2 e^{3x}.$$

2. *Solve* $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$

Solution. Given equation is $(D^3 - D^2 - 4D + 4)\, y$

A.E. is $m^3 - m^2 - 4m + 4 = 0$

$$m^2 (m - 1) - 4 (m - 1) = 0$$

$$(m - 1) (m^2 - 4) = 0$$

$$m = 1, m = \pm 2$$

$$m_1 = 1, m_2 = 2, m_3 = -$$

$\therefore$ The general solution of the given equation is

$$y = C_1 e^x + C_2 e^{2x} + C$$

3. *Solve* $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0.$

Solution. The D.E. can be written as

$$(D^2 - D - 6)\, y = 0$$

A.E. is $m^2 - m - 6 = 0$

$\therefore$ $(m - 3) (m + 2) = 0$

$\therefore$ $m = 3, -2$

$\therefore$ The general solution is

$$y = C_1 e^{3x} + C_2 e^{-2x}.$$

4. *Solve* $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0.$

Solution. The D.E. can be written as

$$(D^2 + 8D + 16)\, y = 0$$

A.E. is $m^2 + 8m + 16 = 0$

$\therefore$ $(m + 4)^2 = 0$

$$(m + 4) (m + 4) = 0$$

$$m = -4, -4$$

$\therefore$ The general solution is

$$y = (C_1 + C_2 x)\, e^{-4x}.$$

5. *Solve* $\frac{d^2y}{dx^2} + w^2y = 0.$

Solution. Equation can be written as

$$(D^2 + w^2)\, y = 0$$

A.E. is $m^2 + w^2 = 0$

$$m^2 = -w^2 = w^2 i^2 \quad (i^2 = -1)$$
$$m = \pm w i$$

This is the form $\alpha \pm i\beta$ where $\alpha = 0$, $\beta = w$.

$\therefore$ The general solution is

$$y = e^{0t}(C_1 \cos wt + C_2 \sin wt)$$

$\therefore$ $y = C_1 \cos wt + C_2 \sin wt$.

6. *Solve* $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 13y = 0.$

Solution. The equation can be written as

$$(D^2 + 4D + 13)\,y = 0$$

A.E. is $\quad m^2 + 4m + 13 = 0$

$$m = \frac{-4 \pm \sqrt{16-52}}{2}$$
$$= -2 \pm 3i \text{ (of the form } \alpha \pm i\beta)$$

$\therefore$ The general solution is

$$y = e^{-2x}(C_1 \cos 3x + C_2 \sin 3x).$$

INVERSE DIFFERENTIAL OPERATOR AND PARTICULAR INTEGRAL

Consider a differential equation

$$f(D)\,y = x \qquad ...(1)$$

Define $\dfrac{1}{f(D)}$ such that

$$f(D)\left\{\frac{1}{f(D)}\right\}x = x \qquad ...(2)$$

Here $f(D)$ is called the inverse differential operator. Hence from Eqn. (1), we obtain

$$y = \frac{1}{f(D)}x \qquad ...(3)$$

Since this satisfies the Eqn. (1) hence the particular integral of Eqn. (1) is given by Eqn. (3)

Thus, particular Integral (P.I.) = $\dfrac{1}{f(D)}x$

The inverse differential operator $\dfrac{1}{f(D)}$ is linear.

i.e., $\quad \dfrac{1}{f(D)}\{ax_1 + bx_2\} = a\dfrac{1}{f(D)}x_1 + b\dfrac{1}{f(D)}x_2$

where a, b are constants and x_1 and x_2 are some functions of x.

SPECIAL FORMS OF THE PARTICULAR INTEGRAL

Type 1: P.I. of the form $\frac{e^{ax}}{f(D)}$

We have the equation $f(D)\, y = e^{ax}$

Let $$f(D) = D^2 + a_1 D + a_2$$

We have $$D(e^{ax}) = a\, e^{ax},\ D^2(e^{ax}) = a^2 e^{ax} \text{ and so on.}$$

$$\therefore \quad f(D)\, e^{ax} = (D^2 + a_1 D + a_2)\, e^{ax}$$
$$= a^2 e^{ax} + a_1 . a e^{ax} + a_2 e^{ax}$$
$$= (a^2 + a_1 . a + a_2)\, e^{ax} = f(a)\, e^{ax}$$

Thus $f(b)\, e^{ax} = f(a)\, e^{ax}$

Operating with $\frac{1}{f(D)}$ on both sides

We get, $$e^{ax} = f(a) . \frac{1}{f(D)} . e^{ax}$$

or $$\text{P.I.} = \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(D)}$$

In particular if $f(D) = D - a$, then using the general formula.

We get, $$\frac{1}{D-a} e^{ax} = \frac{e^{ax}}{(D-a)\phi(D)} = \frac{1}{D-a} . \frac{e^{ax}}{\phi(a)}$$

i.e., $$\frac{e^{ax}}{f(D)} = \frac{1}{\phi(a)} e^{ax} \int 1 . d\,x = \frac{1}{\phi(a)} . x\, e^{ax} \quad \ldots(1)$$

$$\therefore \quad f'(a) = 0 + \phi(a)$$

or $$f'(a) = \phi(a)$$

Thus, Eqn. (1) becomes

$$\frac{e^{ax}}{f(D)} = x . \frac{e^{ax}}{f'(D)}$$

where $$f(a) = 0$$

and $$f'(a) \neq 0$$

This result can be extended further also if

$$f'(a) = 0,\ \frac{e^{ax}}{f(D)} = x^2 . \frac{e^{ax}}{f''(a)} \text{ and so on.}$$

Type 2: P.I. of the form $\frac{\sin ax}{f(D)}, \frac{\cos ax}{f(D)}$

We have $$D(\sin ax) = a \cos ax$$

$$D^2 (\sin ax) = -a^2 \sin ax$$
$$D^3 (\sin ax) = -a^3 \cos ax$$
$$D^4 (\sin ax) = a^4 \sin ax$$
$$= (-a^2)^2 \sin ax \text{ and so on.}$$

Therefore, if $f(D^2)$ is a rational integral function of D^2 then $f(D^2) \sin ax = f(-a^2) \sin ax$.

Hence $\frac{1}{f(D^2)}\left\{f(D^2)\sin ax\right\} = \frac{1}{f(D^2)} f(-a^2)\sin ax$

i.e.,
$$\sin ax = f(-a^2) \frac{1}{f(D^2)} \sin ax$$

i.e.,
$$\frac{1}{f(D^2)}\sin ax = \frac{\sin ax}{f(-a^2)}$$

Provided $f(-a^2) \neq 0$ ...(1)

Similarly, we can prove that

$$\frac{1}{f(D^2)}\cos ax = \frac{\cos ax}{f(-a^2)}$$

if $f(-a^2) \neq 0$

In general, $\frac{1}{f(D^2)}\cos ax = \frac{\cos ax}{f(-a^2)}$

if $f(-a^2) \neq 0$ ...(2)

$$\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b)$$

and
$$\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b)$$

These formula can be easily remembered as follows.

$$\frac{1}{D^2+a^2}\sin ax = \frac{x}{2}\int \sin ax\, dx = \frac{-x}{2a}\cos ax$$

$$\frac{1}{D^2+a^2}\cos ax = \frac{x}{2}\int \cos ax\, ax = \frac{x}{2a}\sin ax.$$

Type 3: P.I. of the form $\frac{\phi(x)}{f(D)}$ where $\phi(x)$ is a polynomial in x, we seeking the polynomial Eqn. as the particular solution of

$$f(D)y = \phi(x)$$

where
$$\phi(x) = a_0 x^n + a_1 x^{n-1} + \dots a_{n-1} x + a_n$$

Hence P.I. is found by divisor. By writing $\phi(x)$ in descending powers of x and $f(D)$ in ascending powers of D. The division get completed without any remainder. The quotient so obtained in the process of division will be particular integral.

Type 1

1. *Solve* $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{5x}$.

Solution. We have

$$(D^2 - 5D + 6)\, y = e^{5x}$$

A.E. is $m^2 - 5m + 6 = 0$

i.e., $(m - 2)(m - 3) = 0$

$\Rightarrow$ $m = 2, 3$

Hence the complementary function is

$\therefore$ $\text{C.F.} = C_1 e^{2x} + C_2 e^{3x}$

Particular Integral (P.I.) is

$$\text{P.I.} = \frac{1}{D^2 - 5D + 6} e^{5x} \qquad (D \to 5)$$

$$= \frac{1}{5^2 - 5\times 5 + 6} e^{5x} = \frac{e^{5x}}{6}.$$

$\therefore$ The general solution is given by

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1 e^{2x} + C_2 e^{3x} + \frac{e^{5x}}{6}.$$

2. *Solve* $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 10e^{3x}$.

Solution. We have

$$(D^2 - 3D + 2)\, y = 10\, e^{3x}$$

A.E. is $m^2 - 3m + 2 = 0$

i.e., $(m - 2)(m - 1) = 0$

$$m = 2, 1$$

$$\text{C.F.} = C_1 e^{2x} + C_2 e^{x}$$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2} 10e^{3x} \qquad (D \to 3)$$

$$= \frac{1}{3^2 - 3\times 3 + 2} 10e^{3x}$$

$$\text{P.I.} = \frac{10\, e^{3x}}{2}$$

$\therefore$ The general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1 e^{2x} + C_2 e^{x} + \frac{10\, e^{3x}}{2}.$$

Type2:

1. *Solve $(D^3 + D^2 - D - 1)\, y = \cos 2x$.*

Solution. The A.E. is

$$m^3 + m^2 - m - 1 = 0$$

$$i.e., \; m^2(m + 1) - 1\,(m + 1) = 0$$

$$(m + 1)\,(m^2 - 1) = 0$$

$$m = -1,\; m^2 = 1$$

$$m = -1,\; m = \pm 1$$

$$\therefore \quad m = -1, -1, 1$$

$$\text{C.F.} = C_1 e^x + (C_2 + C_3 x)\, e^{-x}$$

$$\text{P.I.} = \frac{1}{D^3 + D^2 - D - 1} \cos 2x \qquad (D^2 \to -2^2)$$

$$= \frac{1}{(D+1)(D^2 - 1)} \cos 2x$$

$$= \frac{1}{(D+1)(-2^2 - 1)} \cos 2x$$

$$= \frac{-1}{5}\, \frac{1}{D+1} \cos 2x$$

$$= \frac{-1}{5}\, \frac{\cos 2x}{D+1} \times \frac{D-1}{D-1}$$

$$= \frac{-1}{5}\, \frac{(D-1)\cos 2x}{D^2 - 1} \qquad (D^2 \to -2^2)$$

$$= \frac{-1}{5}\left[\frac{-2\sin 2x - \cos 2x}{-2^2 - 1}\right]$$

$$= \frac{-1}{25}\,(2\sin 2x + \cos 2x)$$

$\therefore$ The general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1 e^x + (C_2 + C_3 x)\, e^{-x} - \frac{1}{25}\,(2\sin 2x + \cos 2x).$$

2. *Solve* $(D^2 + D + 1)\, y = \sin 2x.$

Solution. The A.E. is

$$m^2 + m + 1 = 0$$

i.e.,
$$m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

Hence the C.F. is

$$\text{C.F.} = e^{-\frac{x}{2}} \left[C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x \right]$$

$$\text{P.I.} = \frac{1}{D^2 + D + 1} \sin 2x \qquad (D^2 \to -2$$

$$= \frac{1}{-2^2 + D + 1} \sin 2x$$

$$= \frac{1}{D-3} \sin 2x$$

Multiplying and dividing by $(D + 3)$

$$= \frac{(D+3)\sin 2x}{D^2 - 9}$$

$$= \frac{(D+3)\sin 2x}{-2^2 - 9} = \frac{-1}{13}(2\cos 2x + 3 \sin 2x)$$

$$\therefore\ y = \text{C.F.} + \text{P.I.} = e^{\frac{-x}{2}} \left[C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x \right] - \frac{1}{3}\ (2 \cos 2x + 3 \sin 2x).$$

3. *Solve* $(D^2 + 5D + 6)\, y = \cos x + e^{-2x}.$

Solution. The A.E. is

$$m^2 + 5m + 6 = 0$$

i.e.,
$$(m + 2)(m + 3) = 0$$

$$m = -2, -3$$

$$\text{C.F.} = C_1 e^{-2x} + C_2 e^{-3x}$$

$$\text{P.I.} = \frac{1}{D^2 + 5D + 6} \cdot [\cos x + e^{-2x}]$$

$$= \frac{\cos x}{D^2 + 5D + 6} + \frac{e^{-2x}}{D^2 + 5D + 6}$$

$$= \text{P.I.}_1 + \text{P.I.}_2$$

$$\text{P.I.}_1 = \frac{\cos x}{D^2 + 5D + 6} \qquad (D^2 = -1^2)$$

$$= \frac{\cos x}{-1^2 + 5D + 6} = \frac{\cos x}{5D + 5}$$

$$= \frac{1}{5} \frac{\cos x (D-1)}{(D+1)(D-1)}$$

$$= \frac{1}{5} \frac{(D-1)\cos x}{D^2-1}$$

$$= \frac{1}{5} \frac{-\sin x - \cos x}{-1^2-1}$$

$$= \frac{-1}{5} \frac{\sin x + \cos x}{-2}$$

$$= \frac{1}{10}(\sin x + \cos x)$$

$$\text{P.I.}_2 = \frac{e^{-2x}}{D^2+5D+6} \qquad (D \to -2)$$

$$= \frac{e^{-2x}}{(-2)^2+5\times-2+6} \qquad (Dr = 0)$$

Differential and multiply 'x'

$$= \frac{x\, e^{-2x}}{2D+5} \qquad (D \to -2)$$

$$= \frac{x\, e^{-2x}}{2(-2)+5} = \frac{x\, e^{-2x}}{1} = x\, e^{-2x}$$

$$\text{P.I.} = \frac{1}{10} (\sin x + \cos x) + x\, e^{-2x}$$

$\therefore$ The general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = C_1\, e^{-2x} + C_2\, e^{-3x} + \frac{1}{10} (\sin x + \cos x) + x\, e^{-2x}.$$

Type 3

1. *Solve* $y'' + 3y' + 2y = 12x^2$.

Solution. We have $(D^2 + 3D + 2)\, y = 12x^2$

A.E. is $\quad m^2 + 3m + 2 = 0$

i.e., $\quad (m + 1)(m + 2) = 0$

$\Rightarrow \quad m = -1, -2$

$$\text{C.F.} = C_1 e^{-x} + C_2 e^{-2x}$$

$$\text{P.I.} = \frac{12x^2}{D^2 + 3D + 2}$$

We need to divide for obtaining the P.I.

$$\begin{array}{r|l} & 6x^2 - 18x + 21 \\ \hline 2 + 3D + D^2 & 12x^2 \\ & 12x^2 + 36x + 12 \\ \hline & \quad - 36x - 12 \\ & \quad - 36x - 54 \\ \hline & \qquad\qquad 42 \\ & \qquad\qquad 42 \\ \hline & \qquad\qquad 0 \end{array}$$

Note:
$3D(6x^2) = 36x$
$D^2(6x^2) = 12$

Hence, P.I. $= 6x^2 - 18x + 21$

$\therefore$ The general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = C_1 e^{-x} + C_2 e^{-2x} + 6x^2 - 18x + 21.$$

2. *Solve* $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 2x + x^2$.

Solution. We have $(D^2 + 2D + 1)\, y = 2x + x^2$

A.E. is $\quad m^2 + 2m + 1 = 0$

i.e., $\quad (m + 1)^2 = 0$

i.e., $\quad (m + 1)(m + 1) = 0$

$\Rightarrow \quad m = -1, -1$

$$\text{C.F.} = (C_1 + C_2 x)\, e^{-x}$$

$$\text{P.I.} = \frac{2x + x^2}{D^2 + 2D + 1} = \frac{x^2 + 2x}{1 + 2D + D^2}$$

$$
\begin{array}{r|l}
 & x^2 - 2x + 2 \\
\hline
1 + 2D + D^2 & x^2 + 2x \\
 & x^2 + 4x + 2 \\
\hline
 & -2x - 2 \\
 & -2x - 4 \\
\hline
 & 2 \\
 & 2 \\
\hline
 & 0
\end{array}
$$

$\therefore$ $\qquad$ P.I. $= x^2 - 2x + 2$

$\therefore$ $\qquad$ $y = $ C.F. + P.I.

$= (C_1 + C_2 x)\, e^{-x} + (x^2 - 2x + 2).$

$$A' = \frac{-\sin 2x \cdot 4\tan 2x}{2}, \; B' = \frac{-\cos\, 2x \cdot 4\tan 2x}{2}$$

$$A' = \frac{-2\sin^2 2x}{\cos 2x}, \; B' = 2\,\sin\,2x$$

On integrating, we get

$$A = -2\int \frac{\sin^2 2x}{\cos 2x}\,dx\,, B = 2\int \sin 2x\,dx$$

$$= -2\int \frac{1-\cos^2 2x}{\cos 2x}\,dx$$

$$= -2\int \{\sec 2x - \cos 2x\}\,dx$$

$$= -2\left\{\frac{1}{2}\log\left(\sec 2x + \tan 2x\right) - \frac{1}{2}\sin 2x\right\}$$

$$A = -\log\,(\sec\,2x + \tan\,2x) + \sin\,2x + C_1$$

$$B = 2\int \sin 2x\,dx$$

$$= \frac{2\left(-\cos 2x\right)}{2} + C_2$$

$$B = -\cos\,2x + C_2$$

Substituting these values of A and B in Eqn. (1), we get

$$y = C_1\cos\,2x + C_2\sin\,2x - \cos\,2x\,\log\,(\sec\,2x + \tan\,2x)$$

which is the required general solution.

SOLUTION OF CAUCHY'SHOMOGENEOUS LINEAREQUATION ANDLEGENDRE'S LINEAR EQUATION

A linear differential equation of the form

$$x^n\frac{d^n y}{dx^n} + a_1 x^{n-1}\cdot\frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}\,x\cdot\frac{dy}{dx} + a_n y = \phi(x) \qquad ...(1)$$

Where a_1, a_2, a_3 ...a_n are constants and $\phi(x)$ is a function of x is called a homogeneous linear differential equation of order n.

The equation can be transformed into an equation with constant coefficients by changing the independent variable x to z by using the substitution $x = e^z$ or $z = \log x$

Now $\qquad z = \log x \;\Rightarrow\; \frac{dz}{dx} = \frac{1}{x}$

Consider $\quad \dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx} = \dfrac{1}{x} \cdot \dfrac{dy}{dz}$

$\therefore \quad x\dfrac{dy}{dx} = \dfrac{dy}{dz} = Dy$

where $D = \dfrac{d}{dz}$.

Differentiating w.r.t. 'x' we get,

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 1 = \frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

i.e.,

$$x\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\cdot\frac{1}{x} - \frac{dy}{dx}$$

$$= \frac{1}{x}\cdot\frac{d^2y}{dz^2} - \frac{1}{x}\cdot\frac{dy}{dz}$$

i.e.,

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

i.e.,

$$x^2\frac{d^2y}{dx^2} = (D^2 - D)\,y = D\,(D-1)\,y$$

Similarly,

$$x^3\frac{d^3y}{dx^3} = D\,(D-1)\,(D-2)\,y$$

....

....

$$x^n\frac{d^ny}{dx^n} = D\,(D-1)\,...\,(D-n+1)\,y$$

Substituting these values of $x\dfrac{dy}{dx}, x^2\dfrac{d^2y}{dx^2} \cdots\cdots\cdots x^n\dfrac{d^ny}{dx^n}$ in Eqn. (1), it reduces to a linear differential equation with constant coefficient can be solved by the method used earlier.

Also, an equation of the form,

$$(ax+b)^n \cdot \frac{d^ny}{dx^n} + a_1\,(ax+b)^{n-1}\cdot\frac{d^{n-1}y}{dx^{n-1}} + \ldots any = (x) \qquad ...(2)$$

where a_1, a_2 .....a_n are constants and ϕ (x) is a function of x is called a homogeneous linear differential equation of order n. It is also called "Legendre's linear differential equation".

This equation can be reduced to a linear differential equation with constant coefficients by using the substitution.

$$ax + b = e^z \text{ or } z = \log\,(ax+b)$$

As above we can prove that

$$(ax+b)\cdot\frac{dy}{dx} = a\,Dy$$

$$(ax+b)^2 \cdot \frac{d^2y}{dx^2} = a^2 D (D-1) y$$

....

....

$$(ax+b)^n \cdot \frac{d^ny}{dx^n} = a^n D (D-1)(D-2) \ldots\ldots (D-n+1) y$$

The reduced equation can be solved by using the methods of the previous section.

PROBLEMS:

1. *Solve* $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$.

Solution. The given equation is

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4 \qquad \ldots(1)$$

Substitute $x = e^z$ or $z = \log x$

So that $x\frac{dy}{dx} = Dy, \quad x^2\frac{d^2y}{dx^2} = D(D-1)y$

The given equation reduces to

$$D(D-1)y - 2Dy - 4y = (e^z)^4$$

$$[D(D-1) - 2D - 4]y = e^{4z}$$

i.e., $(D^2 - 3D - 4)y = e^{4z} \qquad \ldots(2)$

which is an equation with constant coefficients

A.E. is $m^2 - 3m - 4 = 0$

i.e., $(m-4)(m+1) = 0$

$\therefore$ $m = 4, -1$

C.F. is C.F. $= C_1e^{4z} + C_2e^{-z}$

$$\text{P.I.} = \frac{1}{D^2-3D-4}e^{4z} \qquad D \to 4$$

$$= \frac{1}{(4)^2-3(4)-4}e^{4z} \qquad Dr = 0$$

$$= \frac{1}{2D-3}ze^{4z} \qquad D \to 4$$

$$= \frac{1}{(2)(4)-3}ze^{4z}$$

$$= \frac{1}{5}ze^{4z}$$

$\therefore$ The general solution of (2) is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = C_1 e^{4z} + C_2 e^{-z} + \frac{1}{5} z e^{4z}$$

Substituting $e^z = x$ or $z = \log x$, we get

$$y = C_1 x^4 + C_2 x^{-1} + \frac{1}{5} \log x \left(x^4\right)$$

$$y = C_1 x^4 + \frac{C_2}{x} + \frac{x^4}{5} \log x$$

is the general solution of the Eqn. (1).

2. *Solve* $x^2 \dfrac{d^2 y}{dx^2} - 3x \dfrac{dy}{dx} + 4y = (x + 1)^2$.

Solution. The given equation is

$$x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 4y = (x + 1)^2 \qquad \text{...(1)}$$

Substituting $\quad x = e^z \quad$ or $\quad z = \log x$

Then $\quad x \dfrac{dy}{dx} = Dy, \quad x^2 \dfrac{d^2 y}{dx^2} = D(D - 1)y$

$\therefore$ Eqn. (1) reduces to

$$D(D - 1)y - 3Dy + 4y = (e^z + 1)^2$$

i.e., $\quad (D^2 - 4D + 4)y = e^{2z} + 2e^z + 1$

which is a linear equation with constant coefficients.

A.E. is $\quad m^2 - 4m + 4 = 0$

i.e., $\quad (m - 2)^2 = 0$

$\therefore \quad m = 2, 2$

$$\text{C.F.} = (C_1 + C_2 z)\, e^{2z}$$

$$\text{P.I.} = \frac{1}{(D-2)^2}\left(e^{2z} + 2e^z + 1\right) \qquad \text{...(2)}$$

$$= \frac{e^{2z}}{(D-2)^2} + \frac{2e^z}{(D-2)^2} + \frac{e^{0z}}{(D-2)^2}$$

$$= \text{P.I.}_1 + \text{P.I.}_2 + \text{P.I.}_3$$

$$\text{P.I.}_1 = \frac{e^{2z}}{(D-2)^2} \qquad (D \to 2)$$

$$= \frac{e^{2z}}{(2-2)^2} \qquad (Dr = 0)$$

$$= \frac{z e^{2z}}{2(D-2)} \qquad (D \to 2)$$

$$= \frac{ze^{2z}}{2(2-2)} \qquad (Dr = 0)$$

$$\text{P.I.}_1 = \frac{z^2 e^{2z}}{2}$$

$$\text{P.I.}_2 = \frac{2e^z}{(D-2)^2} \qquad (D \to 1)$$

$$= \frac{2e^z}{(-1)^2}$$

$$\text{P.I.}_2 = 2e^z$$

$$\text{P.I.}_3 = \frac{e^{0z}}{(D-2)^2} \qquad (D \to 0)$$

$$= \frac{e^{0z}}{4} = \frac{1}{4}$$

$$\text{P.I.} = \frac{z^2}{2} e^{2z} + 2e^z + \frac{1}{4}$$

The general solution of Eqn. (2) is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = (C_1 + C_2 z) e^{2z} + \frac{z^2 e^{2z}}{2} + 2e^z + \frac{1}{4}$$

Substituting $e^z = x$ or $z = \log x$, we get

$$y = (C_1 + C_2 \log x) x^2 + \frac{x^2 (\log x)^2}{2} + 2x + \frac{1}{4}$$

is the general solution of the equation (1).

3. *Solve* $x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 12y = x^2 \log x$.

Solution. The given Eqn. is

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 12y = x^2 \log x \qquad ...(1)$$

Substituting $x = e^z$ or $z = \log x$, so that

$$x \frac{dy}{dx} = Dy, \quad \text{and} \quad x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

Then Eqn. (1) reduces to

$D(D-1)y + 2Dy - 12y = e^{2z}z$

i.e., $(D^2 + D - 12)y = ze^{2z}$ ...(2)

which is the Linear differential equation with constant coefficients.

A.E. is $m^2 + m - 12 = 0$

i.e., $(m + 4)(m - 3) = 0$

$\therefore$ $m = -4, 3$

$$\text{C.F.} = C_1e^{-4z} + C_2e^{3z}$$

$$\text{P.I.} = \frac{1}{D^2 + D - 12} ze^{2z}$$

$$= e^{2z} \frac{z}{(D+2)^2 + (D+2) - 12} \qquad (D \to D + 2)$$

$$= e^{2z} \left[\frac{z}{D^2 + 5D - 6}\right]$$

$$\begin{array}{r|l} & -\frac{1}{6}z - \frac{5}{36} \\ \hline -6 + 5D + D^2 & z \\ & z - \frac{5}{6} \\ \hline & \frac{5}{6} \\ & \frac{5}{6} \\ \hline & 0 \end{array}$$

$$\text{P.I.} = e^{2z}\left[-\frac{z}{6} - \frac{5}{36}\right] = \frac{-e^{2z}}{6}\left[z + \frac{5}{6}\right]$$

$\therefore$ General solution of Eqn. (2) is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = C_1e^{-4z} + C_2e^{3z} - \frac{e^{2z}}{6}\left(z + \frac{5}{6}\right)$$

Substituting $e^z = x$ or $z = \log x$, we get

$$y = C_1x^{-4} + C_2x^3 - \frac{x^2}{6}\left(\log x + \frac{5}{6}\right)$$

$$y = \frac{C_1}{x^4} + C_2x^3 - \frac{x^2}{6}\left(\log x + \frac{5}{6}\right)$$

which is the general solution of Eqn. (1).

Application problems related to higher order differential equations.

1. A particle undergoes forced vibrations according to the law $x''(t) + 25\,x(t) = 21\cos 2t$. If the particle starts from rest at t = 0. Find the displacement at any time t > 0.

Given $x''(t) + 25\,x(t) = 21\cos 2t$.

$(D^2 + 25)x = 21\cos 2t$ where $D = \frac{d}{dt}$

Secondary equation is given $m^2 + 25 = 0$

$$m = \pm 5i$$

$$x_c = (c_1\cos 5t + c_2\sin 5t)$$

$$x_p = 21\cos 2t/(D^2 + 25)$$

Type 2 replace D^2 by -4

$$x_p = 21\cos 2t/21$$

$$x_p = \cos 2t$$

Complete solution is given by $x = x_c + x_p$

$$x = c_1\cos 5t + c_2\sin 5t + \cos 2t \;\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$x'(t) = -5c_1\sin 5t + 5c_2\cos 5t - 2\sin 2t \;\ldots\ldots (2)$$

Given $x = 0$ and $x' = 0$ when $t = 0$

equation 1 reduces to

$$0 = c_1 + 0 + 1, \quad c_1 = -1$$

equation 2 reduces to

$$0 = 0 + 5c_2 - 0, \quad c_2 = 0$$

Using c_1 and c_2 we get $x = -\cos 5t + \cos 2t$

2. A particle moves along the x – axis according to $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 25x = 0$. If the particle is started at x = 0 with an initial velocity of 12 feet/seconds to left determine x(t).

Given $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 25x = 0$ and $\frac{dx}{dt} = -12$

$(D^2 + 6D + 25)\,x = 0$ where $D = \frac{d}{dt}$

Secondary equation is given $m^2 + 6m + 25 = 0$

$m = 3 \pm 4i$

$x = e^{-3t}(c_1\cos 4t + c_2\sin 4t)$........... (1)

Differentiating with respect to t we get

$\frac{dx}{dt} = e^{-3t}(-4c_1\sin 4t + 4c_2\cos 4t) - 3e^{-3t}(c_1\cos 4t + c_2\sin 4t)$................ (2)

Given x = 0 and t = 0 equation 1 reduces to

$0 = 1(c_1 + 0)$

$c_1 = 0$

Given x = 0 and t = 0 and $\frac{dx}{dt} = -12$ equation 2 reduces to

$-12 = 1(0 + 4c_2) - 3(c_1 + 0)$ but c1 = 0

$-12 = 4c_2$

$c_2 = -3$

using c_1 and c_2 equation 1 becomes

$x = x(t) = e^{-3t}(0 - 3\sin 4t)$

Module-2 - Statistical Methods & Curve Fitting

- **Statistical Methods:** Correlation and regression-Karl Pearson's coefficient of correlation and rank correlation -problems. Lines of regression, Angle between regression lines, Regression analysis- lines of regression –problems.

- **Curve Fitting:** Curve fitting by the method of least squares- fitting the curves of the form- $y = ax + b$, $y = ax^2 + bx + c$, $y = ae^{bx}$, $y = ax^b$

Introduction

Correlation

So far, while explain measures of central tendency as well as measures of dispersion, one do the analysis of observations on a single variable or univariate say x.

There are many phenomena where the changes in one variable are related to the changes in the other variable. Suppose two variables x and y are related in such a way that an increase in one is accompanied by an increase or decrease in the other. Such a relationship is called Change in one variable followed by change in other variable is called correlation or covariation

Positive Correlation

A positive correlation is a relationship between two variables where if one variable increases(or decreases), the other one also increases(or decreases)

Example

Demand and price of a commodity are positively correlated as increase in one

results in increase in the other or vice versa.

Negative Correlation or inverse correlation:

A negative correlation is a relationship between two variables where if one variable increases(or decreases), the other one decreases(or increases)

Example

Supply and price of a commodity are negatively correlated or inversely correlated

as increase in one results in decrease in the other or vice versa.

Uncorrelation

If there is no relationship indicated between two variables, are said to be uncorrelated or independent.

Correlation and Regression

This topic deals with data concerning independent observations.

Examples:

1. Marks of individuals in two subjects
2. Height and weight of individuals
3. Amount involved in advertising a product and sale of product etc.,
   We discuss the aspect of inter – relation between the independent variables.

- ## Correlation and Correlation Coefficient

The numerical measure of correlation between two variables x and y is known as

Pearson's coefficient of correlation usually denoted by r and is defined as follows.

$$r = \frac{\sum_1^n (x-\bar{x})(y-\bar{y})}{n\sigma_x\sigma_y} \qquad \dots (1)$$

This can be put in the alternative form as follows.
If X = x - $\bar{x}$, Y = y-$\bar{y}$ we can write

Thus (1) becomes

$$r = \frac{\sum XY}{\sqrt{\sum X^2 \sum Y^2}}$$

Property :

The coefficient of correlation numerically does not exceed unity. ie -1 ≤ r ≤ +1.

Note : If r = ± 1 we say that x and y are perfectly correlated and if r = 0 we say that x and y are non correlated.

- **Alternative formula for the Correlation Coefficient r**

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y}$$

Problems:

1. Calculate the karl – pearson co – efficient for the following ages of husband and wife's.

Roll No.	1	2	3	4	5	6	7	8	9	10
Husband's age (x)	36	23	27	28	28	29	30	31	33	35
Wife's age (y)	29	18	20	22	27	21	29	27	29	28

Soln: $\bar{x} = \frac{\sum X}{10} = \frac{300}{10} = 30$ $\quad \bar{y} = \frac{\sum Y}{10} = \frac{250}{10} = 25$ Here n = 10

x	y	X= x - $\bar{x}$	Y = y - $\bar{y}$	XY	X^2	Y^2
36	29	-7	-7	49	49	49
23	18	-3	-5	15	9	25
27	20	-2	-3	6	4	9
28	22	-2	2	-4	4	4
28	27	-1	-4	4	1	16
29	21	0	4	0	0	16
30	29	1	2	2	1	4
31	27	3	4	12	9	16
33	29	5	3	15	25	9
35	28	6	4	24	36	16
$\sum x = 300$	$\sum y = 250$			$\sum XY = 123$	$\sum X^2 = 138$	$\sum Y^2 = 164$

We have $r = \frac{\sum XY}{\sqrt{\sum X^2 \sum Y^2}}$, $r = \frac{123}{\sqrt{(138)(164)}} = 0.817$

It is a positive correlation.

2. Obtain the correlation of the following data:

x	10	14	18	22	26	30
y	18	12	24	6	30	36

Soln. We have $r = \frac{\sum XY}{\sqrt{\sum X^2 \sum Y^2}}$ where $X = x - \bar{x}$, $Y = y - \bar{y}$

Here n = 6

$\bar{x} = \frac{\sum X}{10} = \frac{120}{6} = 20$ $\bar{y} = \frac{\sum Y}{10} = \frac{126}{6} = 21$

x	y	$X = x - \bar{x}$	$Y = y - \bar{y}$	XY	X^2	Y^2
10	18	-10	-3	30	100	9
14	18	-6	-9	53	36	81
18	18	-2	3	-6	4	9
22	18	2	-15	-30	4	225
26	18	6	9	54	36	81
30	18	10	15	150	100	225
$\sum x = 120$	$\sum y = 126$			$\sum XY = 252$	$\sum X^2 = 280$	$\sum Y^2 = 630$

We have $r = \frac{252}{\sqrt{(280)(630)}} = 0.6$

It is a positive correlation.

3. Calculate Karl pearson co-efficient of correlation b/w the marks obtained by 8 students in mathematics and statistics :

Statistics	8	10	15	17	20	23	24	25
Mathematics	25	30	32	35	37	40	42	45

Solu: Let statistics = x , Mathematics = y

We have $r = \frac{\sum XY}{\sqrt{\sum X^2 \sum Y^2}}$ where $X = x - \bar{x}$, $Y = y - \bar{y}$

Here n = 8

$\bar{x} = \frac{\sum X}{n} = \frac{142}{8} = 17.75$ $\bar{y} = \frac{\sum Y}{n} = \frac{286}{8} = 35.75$

x	y	X= x - $\bar{x}$	Y = y- $\bar{y}$	XY	X^2	Y^2
8	25	-9.75	-10.75	104.81	95.06	115.56
10	30	-7.75	-5.75	44.56	60.06	33.06
15	32	-2.75	-3.75	10.31	7.56	14.06
17	35	-0.75	-0.75	0.56	0.56	0.56
20	37	2.25	1.25	2.81	5.06	1.56
23	40	5.25	4.25	22.31	27.56	18.06
24	42	6.25	6.25	39.06	39.06	39.06
25	45	7.25	9.25	67.06	56.56	85.56
$\sum x = 17.75$	$\sum y = 35.75$			$\sum XY = 291.4$	$\sum X^2 = 291.480$	$\sum Y^2 = 307.48$

We have $r = \frac{291.48}{\sqrt{(291.48)(307.48)}} = 0.97$

It is a positive correlation.

Regression

Regression is an estimation of one independent variable in terms of the other. If x & y are correlated, the best fitting straight line in the least square sense gives reasonably a good relation between x & y.

The best fitting straight line of the form y = ax + b (x being the independent variable) is called the regression line of y on x & x = ay + b (y being the independent variable) is called the regression line of x on y.

Formulas for line of regression

Let y = ax + b be the equation of the regression line of y on x for a given set of n values (x,y).

Then y- $\bar{y}$= $r \frac{\sigma_y}{\sigma_x}$(x-$\bar{x}$)……..............(1)

This is the regression of y on x.

Similarly,

$x - \bar{x} = r \frac{\sigma_x}{\sigma_y}(y - \bar{y})$ …..........(2)

This is the regression line of x on y.

The coefficient of x in (1) & the coefficient of y in (2) respectively given by r $\frac{\sigma_y}{\sigma_x}$ and r $\frac{\sigma_x}{\sigma_y}$ are known as the <u>regression coefficients.</u> Their product is equal to r^2.

Thus we can conclude that r is the geometric mean (GM) of the regression coefficients since the GM of two numbers a, b is $\sqrt{ab}$. That is

$r = \pm\sqrt{(coeff.of\ x)(coeff.of\ y)}$

The sign of r will be positive or negative according as the regression coefficients are positive or negative.

Note: The lines of regression (1) & (2) are also of the form

$$Y = \frac{\sum XY}{\sum X^2}(X) \quad \text{and} \quad X = \frac{\sum XY}{\sum Y^2}(Y) \qquad \text{Where } X = x - \bar{x} \ \ \& \ Y = y - \bar{y}.$$

This form will be useful to find out the coefficient of correction by first obtaining the lines of regression as we have deduced that

$$r = \pm\sqrt{(coeff.of\ x)(coeff.of\ y)}$$

Problems:

1. Compute the coefficient of correlation & the equation of the lines of regression for the following data.

x	1	2	3	4	5	6	7
y	9	8	10	12	11	13	14

Solu: We have coefficient of correlation $r = \frac{\sum XY}{\sqrt{\sum X^2 \sum Y^2}}$

Where $Y = y - \bar{y}$ $\quad X = x - \bar{x}$ $\quad$ Here n = 7

$\bar{x} = \frac{\sum x}{n}$ = 28/7 = 4 $\quad \bar{y} = \frac{\sum y}{n}$ = 77/7 = 11

X	y	X= x- $\bar{x}$	Y= y-$\bar{y}$	XY	X^2	Y^2
1	9	-3	-2	6	9	4
2	8	-2	-3	6	4	9
3	10	-1	-1	1	1	1
4	12	0	1	0	0	1
5	11	1	0	0	1	0
6	13	2	2	4	4	4
7	14	3	3	9	9	9
$\sum x = 28$	$\sum y = 77$			$\sum XY = 26$	$\sum X^2$=28	$\sum Y^2$=28

Line regression y on x

$Y = \frac{\sum XY}{\sum X^2}. X$

$y-\bar{y} = \frac{26}{28}(x- \bar{x})$

y-11 = 0.928 (x-4)

y = 0.92x + 7.29

Line regression x on y

$X = \frac{\sum XY}{\sum Y^2}. Y$

$x-4 = \frac{26}{28}$ (y-11)

x-4 = 0.928y -10.208

x = 0.928y-6.208

2. Obtain the lines of regression and hence find the co-efficient of correlation for the following data

x	1	3	4	2	5	8	9	10	13	15
y	8	6	10	8	12	16	16	10	32	32

Solution:

$\bar{x} = \frac{\sum x}{n}$ = 70/10 = 7 $\bar{y} = \frac{\sum y}{n}$ = 150/10 =15

x	y	X= x- $\bar{x}$	Y= y-$\bar{y}$	xy	X^2	Y^2
1	8	-6	-7	42	36	49
3	6	-4	-9	35	16	81
4	10	-3	-5	15	9	25
2	8	-5	-7	35	25	49
5	12	-2	-3	6	4	9
8	16	1	1	1	1	1
9	16	2	1	2	4	1
10	10	3	-5	-15	9	25
13	32	6	17	102	36	289
15	32	8	17	136	64	289
$\sum x = 10$	$\sum y = 150$			$\sum XY = 360$	$\sum X^2 = 204$	$\sum Y^2 = 818$

Line of regression y on x

$Y = \frac{\sum XY}{\sum X^2}. X$

$y-\bar{y} = 360/204(x-\bar{x})$

y-15 = 1.764 (x-7)

y-15 = 1.764x-12.348

y = 1.764x-12.348+15

y = 1.764x + 2.652

line of regression x on y

$X = \frac{\sum XY}{\sum Y^2}. Y$

$x - 7 = \frac{360}{818}$ (y-15)

x- 7 = 0.44y + 6.6

x = 0.44y + 0.4

We have Co-efficient of correlation $r = \pm\sqrt{(coeff.of\ x)(coeff.of\ y)}$ = 0.88

2. Compute the coefficient of correlation & the equation of the lines of regression for the following data.

x	10	14	18	22	26	30
y	18	12	24	6	30	36

Soln: $\bar{x} = \frac{\sum x}{n} = \frac{120}{6} = 20$ $\bar{y} = \frac{\sum y}{n} = \frac{126}{6} = 21$

x	y	$X = x-\bar{x}$	$Y = y-\bar{y}$	XY	X^2	Y^2
10	18	-10	-3	30	100	9
14	12	-6	-9	54	36	81
18	24	-2	3	-6	4	9
22	6	2	-15	-30	4	225
26	30	6	9	54	36	81
30	36	10	15	150	100	225
$\sum x = 120$	$\sum y = 126$			$\sum XY = 252$	$\sum X^2$=280	$\sum Y^2$=630

We have, Coefficient of correlation $r = \frac{\sum XY}{\sqrt{\sum X^2 \sum Y^2}}$ = 0.6

Line of regression y on x

$Y = \frac{\sum XY}{\sum X^2}. X$

line of regression x on y

$X = \frac{\sum XY}{\sum Y^2}. Y$

$y-\bar{y}=\frac{252}{280}(x-\bar{x})$ $(x-\bar{x})=\frac{252}{630}(y-\bar{y})$

$y - 21 = 0.9 (x - 20)$ $x - 20 = 0.4 (y - 21)$

$y = 0.9x - 18+21$ $x = 0.4y - 8.4 + 20$

$y = 0.9x + 3$ $x = 0.4y + 11.6$

3. If θ is the acute angle between the lines of regression, then show that $\tan\theta = \frac{\sigma_x \sigma_y}{\sigma_x^2+\sigma_y^2}\left(\frac{1-r^2}{r}\right)$. Explain the significance when r = 0 & r = ± 1.

Solu: W. K. T If is θ acute, the angle between the lines $y = m_1 x + c_1$ and $y = m_2 x + c_2$ is given by

$$\tan\theta = \frac{m_2 - m_1}{1+m_1 m_2}$$

We have the lines of regression ,

$$y-\bar{y}= r\frac{\sigma_y}{\sigma_x}(x-\bar{x}) \quad \ldots. (1)$$

$$\text{and } (x-\bar{x}) = r\frac{\sigma_x}{\sigma_y}(y-\bar{y})$$

We write the second of the equation as $y-\bar{y}= \frac{\sigma_y}{r\sigma_x}(x-\bar{x})$ ….(2)

Slope of (1) and (2) are respectively given by

$m_1 = r\frac{\sigma_y}{\sigma_x}$ and $m_2 = \frac{\sigma_y}{r\sigma_x}$

Substituting these in the formula for tan θ we have,

$$\tan\theta = \frac{r\frac{\sigma_y}{\sigma_x} - \frac{\sigma_y}{r\sigma_x}}{1+r\frac{\sigma_y}{\sigma_x}\frac{\sigma_y}{r\sigma_x}}$$

$$\tan\theta = \frac{\sigma_x \sigma_y}{\sigma_x^2+\sigma_y^2}\left(\frac{1-r^2}{r}\right)$$

If r = ± 1 , $\tan\theta = 0 \rightarrow \theta = 0$,which implies that the two regression lines coincide and hence the variables are perfectely correlated. Also if r = 0 , $\tan\theta = \infty$ or $\theta = \frac{\pi}{2}$. This implies that the lines are perpendicular and hence the variables are uncorrelated.In a partially destroyed record, only the lines of regression of y on x and x on y are available as 4x – 5y + 33 = 0 and 20x – 9y = 107 respectively. Calculate $\overline{x}$, $\overline{y}$ and the coefficient of correlation between x and y.

Solu. W .K .T regression lines passes through $\bar{x}$ and $\bar{y}$

$$4\bar{x} - 5\bar{y} = -33 \qquad \text{and} \qquad 20\bar{x} - 9\bar{y} = 107$$

By solving we get $\bar{x} = 13$ $\quad \bar{y} = 17$

We shall now rewrite the equation of the regression lines to find the regression coefficients.

5y = 4x + 33 or y = 0.8 x + 6.6 ….(1)

20x = 9y + 107 or x = 0.45y + 4.35 …(2)

$r = \pm\sqrt{(cofficient\ of\ x\ )(coefficient\ of\ y)} = \pm\sqrt{(0.8)(0.45)}$,

r = 0.6

It is a positive correlation

5) In a partially destroyed laboratory data, only the regression lines with equations $3x + 2y = 26$, and $6x + y = 31$ are available. Calculate the means of $x's$, means of $y's$ and the correlation co-efficient.

Solution: Since the regression lines passes through $(\overline{x}, \overline{y})$,

$$3x + 2y = 26 \quad and \quad 6x + y = 31 \ .$$

By solving above two equations we get, $\overline{x} = 4$, $\overline{y} = 7$.

Given regression lines are $y = -\frac{3}{2}x + 13$ and $x = -\frac{1}{6}y + \frac{31}{6}$.

Since co-efficient of correlation is geometric mean between the two regressions coefficients

$$r = \sqrt{-\frac{3}{2} \times -\frac{1}{6}} = -0.5$$

(−ve sign is taken since both the regressions coefficients are −ve)

Rank Correlation and an expression for the rank correlation coefficient.

- The coefficient of correlation in respect of the ranks of some two characteristics of an individual or an observation is called Rank Correlation Coefficient usually denoted by ρ

We now proceed to derive an expression for ρ in the following form.

$$\rho = 1 - \frac{6\sum(x-y)^2}{n\,(n^2-1)} \quad \text{or} \quad 1 - \frac{6\sum d^2}{n\,(n^2-1)}$$

Note:

(1) If the ranking of x ,y are entirely in the same order like for example, x :1 1,2,3,4,5 ; y : 1,2,3,4,5 then $\sum d^2 = \sum(x-y)^2 = 0$. This will give us $\rho = \pm 1$ and is called perfect direct correlation.
If the ranking of x and y are entirely in the opposite order like for example, x : 1,2,3,4,5
y : 5,4,3,2,1 then $\sum d^2 = 40$.This will give us ρ = -1 and is called perfect inverse correlation.

Problems:

1. Ten competitors in a beauty contest are ranked by two judges in the following order. Compute the coefficient of correlation

I	1	6	5	3	10	2	4	9	7	8
II	6	4	9	8	1	2	3	10	5	7

Soln : We have $\rho = 1 - \frac{6\sum d^2}{n\,(n^2-1)}$

For the given data, n = 10 and

$\sum d^2 = (1-6)^2 + (6-4)^2 + (5-9)^2 + (3-8)^2 + (10-1)^2 + (2-2)^2 + (4-3)^2 + (9-10)^2 + (7-5)^2 + (8-7)^2$

= 25 + 4+ 16 + 25 + 81 + 0 +1 +1 + 4 + 1 =158

Hence $\rho = 1 - \frac{6\,(158)}{10(10^2-1)}$ = 0.042Ten students got the following percentage of marks in two subjects x and y. Compute their rank correlation coefficient.

Marks in x	78	36	98	25	75	82	90	62	65	39
Marks in y	84	51	91	60	68	62	86	58	53	47

Soln : We prepare the table consisting of the given data along with the ranks assigned according to their order of the magnitude. In the subject x, 98 will be awarded rank 1, 90 as rank 2 and so on.

Marks in x	Rank(x)	Marks in y	Rank(y)	d = (x-y)	$d^2 = (x-y)^2$
78	4	84	3	1	1
36	9	51	9	0	0
98	1	91	1	0	0
25	10	60	6	4	16
75	5	68	4	1	1
82	3	62	5	-2	4
90	2	86	2	0	0
62	7	58	7	0	0
65	6	53	8	-2	4
39	8	47	10	-2	4
					$\sum d^2 = 30$

We have $\rho = 1 - \frac{6\sum d^2}{n\,(n^2 - 1)}$ and n = 10 for the given data.

$= 1 - \frac{6\,(30)}{10(10^2 - 1)}$

= 0.82

2. Ten competitors in music contest are ranked by 3 judges A, B,C in the following order.

Use the rank correlation coefficient to decide which pair of judges have the nearest

. approach to common taste of music

A	1	6	5	10	3	2	4	9	7	8
B	3	5	8	4	7	10	2	1	6	9
C	6	4	9	8	1	2	3	10	5	7

Soln : We shall compute ρ_{AB} , ρ_{BC} , ρ_{CA} with the help of the following table where d is the difference in ranks.

A	B	C	d^2_{AB}	d^2_{BC}	d^2_{CA}
1	3	6	4	9	25
6	5	4	1	1	4
5	8	9	9	1	16
10	4	8	36	16	4
3	7	1	16	36	4
2	10	2	64	64	0
4	2	3	4	1	1
9	1	10	64	81	1
7	6	5	1	1	4
8	9	7	1	4	1
			$\sum d^2_{AB} = 200$	$\sum d^2_{BC} = 214$	$\sum d^2_{CA} = 60$

We have $\rho = 1 - \frac{6\sum d^2}{n\,(n^2-1)}$ and n = 10 for the given data.

Now, $\rho_{AB} = 1 - \frac{6\,(200)}{10(10^2-1)} = -0.21$, $\rho_{BC} = 1 - \frac{6\,(214)}{10(10^2-1)} = -0.297$

$\rho_{CA} = 1 - \frac{6\,(60)}{10(10^2-1)} = +0.636$

It may be observed that ρ_{AB} and ρ_{BC} are negative which means their tastes (A &B; B &C) are opposite. But ρ_{CA} is positive and is nearer to 1.(perfect correlation)

CURVE FITTING

CONTENTS:

- **Curve fitting by the method of least squares**
- **Fitting of curves of the form**
  - y = ax+ b
  - $y = ax^2 + bx + c$
  - $y = ae^{bx}$

CURVE FITTING

CURVE FITTING [BY THE METHOD OF LEAST SQUARE]:

We can plot 'n' points (x_i, y_i) where i=0,1,2,3,………
At the XY plane. It is difficult to draw a graph y=f(x) which passes through all these points but we can draw a graph which passes through maximum number of point. This curve is called the curve of best fit. The method of finding the curve of best fit is called
the curve fitting.

FITTING A STRAIGHT LINE Y = AX + B:

We have straight line that sounds as best approximate to the actual curve y=f(x) passing through 'n' points (x_i, y_i) , i=0,1.2 ....n equation of a straight line is

$y = a + bx$ (1)

Then for 'n' points (2) i i Y =(a+bx_i).......... (2)

Where a and b are parameters to be determined; Yi is called the estimated value. The given value Yi corresponding to xi .
Normal equations are $\sum y = na + b\sum x$, $\sum xy = a\sum x + b\sum x^2$

Where 'n' is the number of points or value.

FITTING A SECOND DEGREE PARABOLA Y=AX^2 + BX + C:

Let us take equation of parabola called parabola of best fit in the form

y= ax^2 + bx + c
normal equations are

$$\sum y = a\sum x^2 + b\sum x + nc$$
$$\sum xy = a\sum x^3 + b\sum x^2 + c\sum x$$

$$\sum x^2 y = a\sum x^4 + b\sum x^3 + c\sum x^2$$

Fit a curve of the form y=ab^x:

Consider y=ab^x …….(1)

Take log on both side

log y = log(ab^x)

log y =log a + log b^x

∴ Y = A + B x …….(2) ; log y = Y → $y = e^Y$,

log a = A → $a = e^A$

log b = B → $b = e^B$

Corresponding normal equations are

$$\sum Y = nA + B\sum x \quad …..(3)$$

$$\sum xY = A\sum x + B\sum x^2 \quad …..(4)$$

Solving the normal equation (3) & (4) for a & b . Substitute these values in (1) we get curve of best fit of the form y= ab^x

PROBLEMS:

1. **Fit a straight line y=a+bx to the following data**

x:	**5**	**10**	**15**	**20**	**25**
y:	**16**	**19**	**23**	**30**	**26**

Soln : Let y =a + b x ……(1)

Normal equations are $\sum y = na + b\sum x$ …..(2)

$$\sum xy = a\sum x + b\sum x^2 \quad …..(3)$$

x	y	x^2	xy
5	16	25	80
10	19	100	190
15	23	225	345
20	26	400	520
25	30	625	750
$\sum x = 75$	$\sum y = 114$	$\sum x^2 = 1375$	$\sum xy = 1885$

From (2) & (3),

114 = 15a +75b
1885 =75a +1375b
a = 12.3, b = 0.7
Becomes y = 12.3 + 0.7 x

2. **Fit a straight line y=a+bx to the following data**

x:	**1**	**2**	**3**	**4**	**5**
y:	**14**	**13**	**9**	**5**	**2**

Soln : Let y =a + b x ……(1)

Normal equations are $\sum y = na + b\sum x$ …..(2)

x	y	x^2	xy
1	14	1	14
2	13	4	26
3	9	9	27
4	5	16	20
5	2	25	10
$\sum x = 15$	$\sum y = 43$	$\sum x^2 = 55$	$\sum xy = 97$

$\sum xy = a\sum x + b\sum x^2$ …..(3)

From (2) & (3),

43 15a +15b , 97=15a +56b , a = 18.2 ,,b = -3.2 equn(1)Becomes y = 18.2 -3.2 x

3. **Fit a straight line y=a+bx to the following data**

x:	**0**	**1**	**2**	**3**	**4**
y:	**1**	**1.8**	**3.3**	**4.5**	**6.3**

Soln : Let y =a + b x ……(1)

Normal equations are $\sum y = na + b\sum x$ …..(2)

$$\sum xy = a\sum x + b\sum x^2 \quad …..(3)$$

X	y	x^2	xy
0	1	0	0
1	1.8	1	1.8
2	3.3	4	6.6
3	4.5	9	13.5
4	6.3	16	25.2
$\sum x = 10$	$\sum y = 16.9$	$\sum x^2 = 30$	$\sum xy = 47.1$

From (2) & (3),
16.9 = 5a +10b
47.1 =10a +30b
a = 0.72
b = 1.33
equ(1)Becomes y = 0.72 + 1.33 x

4. If p is the pull required to lift a load by means of pulley block. Find a linear block of the form p=MW+C Connected p &w using following data

w:	**50**	**70**	**100**	**120**
p:	**12**	**15**	**21**	**25**

Compute p when W=150.

Soln: Given p=y & W=x
Equation of straight line is y =a + b x ……(1)
Normal equations are $\sum y = na + b\sum x$ …..(2)

$$\sum xy = a\sum x + b\sum x^2 \quad …..(3)$$

x	y	x^2	xy
50	12	2,500	600
70	15	4,900	1,050
100	21	10,000	2,100
120	25	14,400	3,000
$\sum x = 340$	$\sum y = 73$	$\sum x^2 = 31{,}800$	$\sum xy = 6{,}750$

From (2) & (3),
73 = 4a +340b , 6750=340a +31800b
a = 2.27 , b = 0.187
Becomes y = 2.27 + 0.187 x ,
Put w=150 , in (1) y = 30.32

4. Fit a parabola y= $a+bx + cx^2$ for the following data

x:	**1**	**2**	**3**	**4**
Y :	**1.7**	**1.8**	**2.3**	**3.2**

Soln : y= $a+bx + cx^2$ (1)
Normal equations are
$\sum y = na \ + +c\sum x^2$ (2)

$\sum xy = a\sum x + b\sum x^2 + c\sum x^3$ (3)

$\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4$ (4)

x	y	x^2	x^3	x^4	xy	x^2y
1	1.7	1	1	1	1.7	1.7
2	1.8	4	8	16	3.6	7.2
3	2.3	9	27	81	6.9	20.7
4	3.2	16	64	256	12.8	51.2
$\sum x = 10$	$\sum y = 9$	$\sum x^2 = 30$	$\sum x^3 = 100$	$\sum x^4 = 354$	$\sum xy = 25$	$\sum x^2y = 80.8$

From (1) ,(3) & (4)
9=4a+10b+30c
25=10a+30b+100c
80.8=30a+100b+354c
a=2, b=-0.5, c=0.2

(1) Becomes $y = 2 - 0.5x + 0.2(x)^2$

4. **Fit a curve of the form $y=ae^{bx}$ for the following data**

x:	0	2	4
y:	8.12	10	31.82

Soln : Let $y= ae^{bx}$ ……(1)

Normal equations are

$\sum Y = nA + b\sum x$ …..(2) log y = Y → $y = e^{Y}$,
log a = A → $a = e^{A}$

$\sum xY = A\sum x + b\sum x^2$ …..(3)

X	y	Y=logy	x^2	xY
0	8.12	2.093	0	0
2	10	2.302	4	4.604
4	31.82	3.46	16	13..84
$\sum x = 6$		$\sum Y = 7.86$	$\sum x^2 = 20$	$\sum xY = 18.444$

From (2) & (3),7.856 = 3A +6b , 18.444=6A +20b ,
A = 1.935, B=0.341 , Then $e^{A} = a = 6.924$, Becomes $y = 6.924e^{0.341x}$

5. **Fit a II degree parabola y= $ax^2 + bx + c$ to the least square method & find y when x=6**

x:	1	2	3	4	5
y:	10	12	13	16	19

Soln : Let y= ax^2 + bx + c (1)

Normal equations are

$\sum y = a\sum x^2 + b\sum x + nc$ (2)

$\sum xy = a\sum x^3 + b\sum x^2 + c\sum x$ (3) , $\sum x^2y = a\sum x^4 + b\sum x^3 + c\sum x^2$ (4)

x	y	x^2	x^3	x^4	xy	x^2y
1	10	1	1	1	10	10
2	12	4	8	16	24	14
3	13	9	27	81	39	117
`4	16	16	64	256	64	256
5	19	25	125	625	95	475
$\sum x = 15$	$\sum y = 70$	$\sum x^2 = 55$	$\sum x^3 = 225$	$\sum x^4 = 979$	$\sum xy = 232$	$\sum x^2y = 906$

70 =5c+ 15b +55a
232= 15c+ 55b+ 225a
906 =55c+ 225b+ 979a
a=0.285 , b=0.485 , c=9.4

(1) Becomes
y= 0.285x^2+ 0.485x+ 9.4 , at x=6, y=22.6

6. **The revolution r and the time t are related by $r = at^2 + bt + c$, Estimate the number of revolutions for time 3.5 units. Given that,**

Revolution	5	10	15	20	25	30	35
time	1.2	1.6	1.9	2.1	2.4	2.6	3

Sol: Normal equations for the curve $r = at^2 + bt + c$ are

$$a\sum t^4 + b\sum t^3 + c\sum t^2 = \sum t^2 r$$
$$a\sum t^3 + b\sum t^2 + c\sum t = \sum t\, r \qquad \cdots\cdots\cdots (1)$$
$$a\sum t^2 + b\sum t + nc = \sum r$$

From the given data

$n = 7$, $\sum t^4 = 200.9826$, $\sum t^3 = 80.344$, $\sum t^2 = 33.54$, $\sum t = 14.8$

.

$\sum t^2 r = 836.95$, $\sum t\, r = 335.5$, $\sum r = 140.$ $\cdots\cdots\cdots (2)$

Substituting in the normal equations, we get

$$200.9826\ a + 80.344b + 33.54c = 836.95 \qquad a = 0.6646$$
$$80.344a + 33.54b + 14.8c = 335.5 \quad \Rightarrow \quad b = 14.7795$$
$$33.54a + 14.8b + 7c = 140 \qquad c = -14.4322$$

Therefore $r = 0.6646t^2 + 14.7795t - 14.4322$.

And hence $r(3.5) = 45.4374$.

7) Fit a curve $y = ae^{bx}$ for the following data,

x	1	5	7	9	12
y	10	15	12	15	21

Sol: For the curve $y = ae^{bx}$, taking log, $\log y = \log a + bx$

Or $Y = A + bx$. Where $Y = \log y$, $A = \log a$. $\cdots\cdots\cdots (1)$

Normal equations are, $nA + b\sum x = \sum Y$ and $A\sum x + b\sum x^2 = \sum xY$. $\cdots\cdots\cdots (1)$

From the given data, $n = 5$.

$\sum x = 34$. $\sum Y = \sum \log_e y = 13.2481$

$\sum x^2 = 300$. $\sum xY = \sum x \log_e y = 94.1439$. $\cdots\cdots\cdots (2)$

Substituting in the normal equations, we get

$$5A + 34b = 13.2481$$

$$34\,A + 300b = 94.1439.$$

$\Longrightarrow$ $A = 2.2487$, $b = 0.0590$. .

And $a = e^A = 9.4754$. $\therefore$ $y = 9.4754e^{0.0590x}$.

1. Fit a curve of the curve y=ax^b for the data

x:	**1**	**1.5**	**2**	**2.5**
y:	**2.5**	**5.61**	**10.0**	**15.6**

Soln : Let y= ax^b ……(1)

Normal equations are

$\sum Y = nA + B\sum x$ …..(2) log y = Y → $y = e^Y$, log a = A → $a = e^A$

$\sum XY = A\sum X + B\sum X^2$ …..(3) log x = X → $x = e^X$

x	y	X=logx	Y=logy	X^2	XY
1	2.5	0	0.916	0	0
1.5	5.62	0.405	1.726	0.164	0.699
2	10.0	0.693	2.302	0.480	1.595
2.5	15.6	0.916	2.747	0.839	2.516
		$\sum X = 2.014$	$\sum Y = 7.691$	$\sum X^2 = 1.483$	$\sum XY = 4.81$

Module-3 – Fourier Series

- Periodic functions, Dirchlet's condition, conditions for a Fourier series expansion, Fourier series of functions with period 2π and with arbitrary period. Half rang Fourier series. Practical harmonic analysis.
- Application to variation of periodic current.
- **Self-study:** Typical waveforms, complex form of Fourier series

FOURIER SERIES

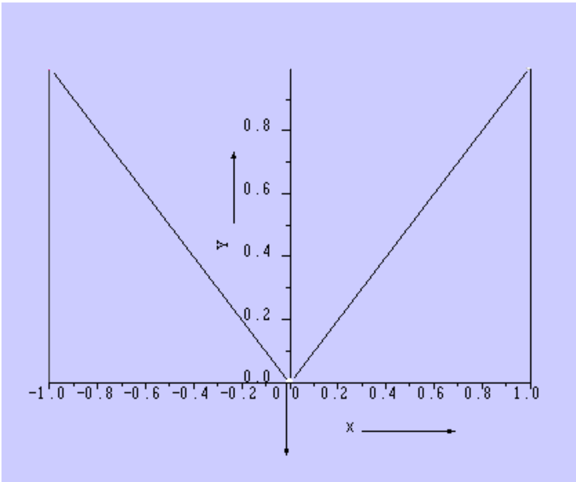
Introduction: In many engineering problems, especially in the study of periodic phenomenae in conduction of heat, electro-dynamics and acoustics, it is necessary to express a function in a series of sines and cosines. Such a series is known as the **Fourier series.**

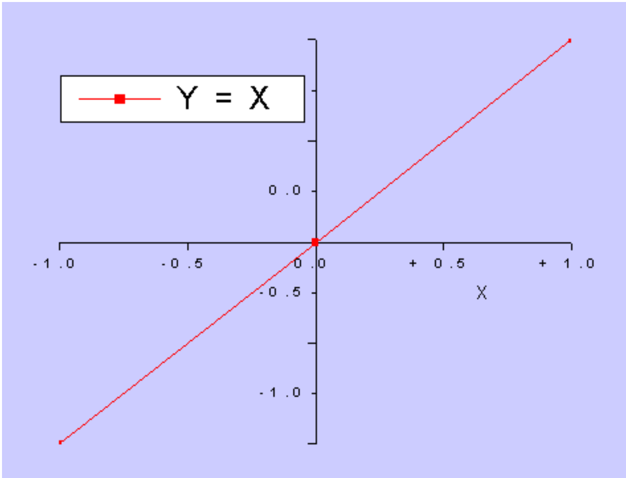
DEFINITIONS:

A function $y = f(x)$ is said to be even, if $f(-x) = f(x)$. The graph of the even function is always symmetrical about the y-axis.

A function $y = f(x)$ is said to be odd, if $f(-x) = -f(x)$. The graph of the odd function is always symmetrical about the origin.

For example, the function f(x) = $|x|$ in [-1, 1] is even as f(-x) = $|-x| = |x|$ = f(x) and the function f(x) = x in [-1, 1] is odd as f(-x) = -x = -f(x). The graphs of these functions are shown below:





Graph of f(x) = $|x|$ Graph of f(x) = x

Note that the graph of f(x) = $|x|$ is symmetrical about the y-axis and the graph of f(x) = x is symmetrical about the origin.

1. If f(x) is even and g(x) is odd, then
   - h(x) = f(x) x g(x) is odd
   - h(x) = f(x) x f(x) is even
   - h(x) = g(x) x g(x) is even

For example,

1. $h(x) = x^2 \cos x$ is even, since both x^2 and $\cos x$ are even functions
2. $h(x) = x \sin x$ is even, since x and $\sin x$ are odd functions
3. $h(x) = x^2 \sin x$ is odd, since x^2 is even and $\sin x$ is odd.

2. If f(x) is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

3. If f(x) is odd, then $\int_{-a}^{a} f(x)dx = 0$

For example,

$$\int_{-a}^{a} \cos x dx = 2\int_{0}^{a} \cos x dx, \text{ as } \cos x \text{ is even}$$

and $\int_{-a}^{a} \sin x dx = 0,$ as $\sin x$ is odd

<u>PERIODIC FUNCTIONS</u>:-

A function $f(x)$ is said to be periodic function with period T if $f(x+T) = f(x)$

Here f(x) is a real-valued function and T is a positive real number.

As a consequence, it follows that

$$f(x) = f(x+T) = f(x+2T) = f(x+3T) = \dots\dots\dots. = f(x+nT)$$

Thus, $f(x) = f(x+nT), n = 1, 2, 3, 4, \dots\dots\dots$

The function $f(x) = \sin x$ is periodic of period 2π since

$$\sin(x + 2n\pi) = \sin x, \quad n=1,2,3,\dots\dots$$

Note that the graph of the function between 0 and 2π is the same as that between 2π and 4π and so on. It may be verified that a linear combination of periodic functions having period T is also periodic of period T.

<u>EULER'S FORMULAE</u>:

The Fourier series for the function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)dx, \; a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx dx, \; b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx dx,$

These values of a_0, a_n, b_n are known as Euler's formulae.

Proof: Let $f(x)$ be represented in the interval $(\alpha, \alpha + 2\pi)$ by the Fourier series-

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \text{ ---------- (1)}$$

To find a_0, a_n, b_n, assume that the series (1) can be integrated term by term from $x=\alpha$ to $x=\alpha+2\pi$.

To find a_0: $\displaystyle\int_{\alpha}^{\alpha+2\pi} f(x)dx=\frac{1}{2}a_0\int_{\alpha}^{\alpha+2\pi}dx+\int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty}a_n\cos nx\right)dx+\int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty}b_n\sin nx\right)dx$

$$=\frac{1}{2}a_0.(\alpha+2\pi-\alpha)+0+0=a_0\pi$$

Since $\displaystyle\int_{\alpha}^{\alpha+2\pi}\cos nxdx=\frac{\sin nx}{n}\Bigg|_{\alpha}^{\alpha+2\pi}=0$ & $\displaystyle\int_{\alpha}^{\alpha+2\pi}\sin nxdx=-\frac{\cos nx}{n}\Bigg|_{\alpha}^{\alpha+2\pi}=0$

$$\therefore\ a_0=\frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi}f(x)dx$$

To find a_n, multiply each side of (1) by cosnx and integrate from $x=\alpha$ to $x=\alpha+2\pi$, we get

$$\int_{\alpha}^{\alpha+2\pi} f(x)\cos nxdx=\frac{1}{2}a_0\int_{\alpha}^{\alpha+2\pi}\cos nxdx+\int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty}a_n\cos nx\right)\cos nxdx+\int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty}b_n\sin nx\right)\cos nxdx$$

$$=0+\pi a_n+0=\pi a_n$$

Since

$$\int_{\alpha}^{\alpha+2\pi}\cos^2 nxdx=\int_{\alpha}^{\alpha+2\pi}\frac{1+\cos 2nx}{2}dx=\frac{1}{2}\left(x+\frac{\sin 2nx}{2n}\right)\Bigg|_{\alpha}^{\alpha+2\pi}=\frac{1}{2}\left(\alpha+2\pi-\alpha+0-0\right)=\pi$$

$$\& \int_{\alpha}^{\alpha+2\pi}\sin nx\cos nxdx=\frac{\sin^2 nx}{2n}\Bigg|_{\alpha}^{\alpha+2\pi}=0$$

Hence $\displaystyle a_n=\frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi}f(x)\cos nxdx$

Similarly To find b_n, multiply each side of (1) by sin nx and integrate from $x=\alpha$ to $x=\alpha+2\pi$, we get

$$\int_{\alpha}^{\alpha+2\pi} f(x)\sin nxdx=\frac{1}{2}a_0\int_{\alpha}^{\alpha+2\pi}\sin nxdx+\int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty}a_n\cos nx\right)\sin nxdx+\int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty}b_n\sin nx\right)\sin nxdx$$

$$=0+0+\pi b_n=\pi b_n$$

Since

$$\int_{\alpha}^{\alpha+2\pi}\sin^2 nxdx=\int_{\alpha}^{\alpha+2\pi}\frac{1-\cos 2nx}{2}dx=\frac{1}{2}\left(x-\frac{\sin 2nx}{2n}\right)\Bigg|_{\alpha}^{\alpha+2\pi}=\frac{1}{2}\left(\alpha+2\pi-\alpha+0-0\right)=\pi$$

$$\& \int_{\alpha}^{\alpha+2\pi}\sin nx\cos nxdx=\frac{\sin^2 nx}{2n}\Bigg|_{\alpha}^{\alpha+2\pi}=0$$

Hence $\displaystyle b_n=\frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi}f(x)\sin nxdx$

FOURIER SERIES:

A Fourier series of a periodic function consists of a sum of sine and cosine terms. Sines and cosines are the most fundamental periodic functions.

The Fourier series is named after the French Mathematician and Physicist Jacques Fourier (1768 – 1830).

FORMULA FOR FOURIER SERIES

Consider a real-valued function f(x) which obeys the following conditions called Dirichlet's conditions:

1. f(x) is defined in an interval (a, a+2*l*), and f(x+2*l*) = f(x) so that f(x) is a periodic function of period 2*l*.
2. f(x) is continuous or has only a finite number of discontinuities in the interval (a, a+2*l*).
3. f(x) has no or only a finite number of maxima & minima in the interval (a, a+2*l*).

Also, let $a_0 = \frac{1}{l}\int_a^{a+2l} f(x)dx$ ---------------- (1)

$$a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx,\; n = 1,2,3,..... \quad \text{--------- (2)}$$

$$b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx,\; n = 1,2,3,...... \quad \text{---------- (3)}$$

Then, the infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right)x \qquad \text{---------- (4)}$$

is called the Fourier series of f(x) in the interval (a, a+2*l*). Also, the real numbers a_0, a_1, a_2, ….a_n, and b_1, b_2 , ….b_n are called the Fourier coefficients of f(x). The formulae (1), (2) and (3) are called Euler's formulae.

It can be proved that the sum of the series (4) is f(x) if f(x) is continuous at x. Thus we have f(x)

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}\right)x \,....... \qquad (5)$$

Suppose f(x) is discontinuous at x, then the sum of the series (4) would be

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right]$$

where $f(x^+)\,\&\,f(x^-)$ are respectively right hand and left hand limits of $f(x)$ given by.

$$f(x^+) = \lim_{h\to 0} f(x+h), \quad f(x^-) = \lim_{h\to 0} f(x-h),\; h > 0$$

Particular Cases:

Case (i)

Suppose a = 0. Then f(x) is defined over the interval (0,2*l*). Formulae (1), (2), (3) reduce to

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx, \qquad n = 1,2,......\infty \text{ ----------- (6)}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx,$$

Then the right-hand side of (5) is the Fourier expansion of f(x) over the interval (0, 2*l*).

If we set $l = \pi$, then f(x) is defined over the interval (0, 2π). Formulae (6) reduce to

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx, \quad n=1, 2,.....\infty \quad \text{--------- (7)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx \qquad n=1, 2,.....\infty$$

Also, in this case, (5) becomes: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ------------ (8)

Case (ii)

Suppose a = -*l*. Then f(x) is defined over the interval (-*l*, *l*). Formulae (1), (2) (3) reduce to

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{+l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx, \quad n=1, 2,.....\infty \quad \text{--------- (9)}$$

$$b_n = \frac{1}{l}\int_{-l}^{+l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx \qquad n=1, 2,.....\infty$$

Then the right-hand side of (5) is the Fourier expansion of f(x) over the interval (-*l*, *l*).

If we set $l = \pi$, then f(x) is defined over the interval (-π, π). Formulae (9) reduce to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \qquad n=1,2,\ldots\ldots\infty \quad \text{----------} \quad (10)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \qquad n=1,2,\ldots\ldots\infty$$

Putting $l = \pi$ in (5), we get f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Some useful results:

1.The following rule called Bernoulli's generalized rule of integration by parts is useful in evaluating the Fourier coefficients.

$$\int uv dx = uv_1 - u' v_2 + u'' v_3 + \ldots\ldots.$$

Here u', u'',….. are the successive derivatives of u and $v_1 = \int v dx, v_2 = \int v_1 dx, \ldots\ldots$

We illustrate the rule, through the following examples:

$$\int x^2 \sin nx dx = x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)$$

$$\int x^3 e^{2x} dx = x^3\left(\frac{e^{2x}}{2}\right) - 3x^2\left(\frac{e^{2x}}{4}\right) + 6x\left(\frac{e^{2x}}{8}\right) - 6\left(\frac{e^{2x}}{16}\right)$$

2. The following integrals are also useful:

$$\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$$

$$\int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$$

3. If 'n' is integer, then

$\sin n\pi = 0$, $\cos n\pi = (-1)^n$, $\sin 2n\pi = 0$, $\cos 2n\pi=1$

$\sin(n+1/2)\pi = (-1)^n$, $\cos(n+1/2)\pi = 0$

Examples

1. Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$

We have $a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x} dx = \left.\frac{e^{-x}}{-1}\right|_0^{2\pi} = -\frac{1}{\pi}\left(e^{-2\pi} - 1\right) = \frac{1-e^{2\pi}}{\pi}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx dx = \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-\cos 2n\pi + n\sin 2n\pi) - e^0(-\cos 0 + n\sin 0)\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[-e^{-2\pi}+1\right] = \left(\frac{1-e^{-2\pi}}{\pi}\right).\frac{1}{1+n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx dx = \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(-\sin nx - n\cos nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-\sin 2n\pi - n\cos 2n\pi) - e^0(-\sin 0 - n\cos 0)\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[-ne^{-2\pi}+n\right] = \left(\frac{1-e^{-2\pi}}{\pi}\right).\frac{n}{1+n^2}$$

$$\therefore\ e^{-x} = \frac{1-e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty}\left(\frac{1-e^{-2\pi}}{\pi}.\frac{1}{1+n^2}\right)\cos nx + \sum_{n=1}^{\infty}\left(\frac{1-e^{-2\pi}}{\pi}.\frac{n}{1+n^2}\right)\sin nx$$

$$= \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{1}{1+n^2}\right)\cos nx + \sum_{n=1}^{\infty}\left(\frac{n}{1+n^2}\right)\sin nx\right]$$

2. Obtain the Fourier expansion of f(x) = $\frac{1}{2}(\pi - x)$ in $-\pi < x < \pi$

We have, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)dx = \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_{-\pi}^{\pi} = \pi$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)\cos nx dx$$

Here we use integration by parts, so that

$$a_n = \frac{1}{2\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{\pi} = \frac{1}{2\pi}[0] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)\sin nx dx = \frac{1}{2\pi}\left[(\pi - x)\frac{-\cos nx}{n} - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{\pi} = \frac{(-1)^n}{n}$$

Using the values of a_0 , a_n and b_n in the Fourier expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

we get, $f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$

This is the required Fourier expansion of the given function.

3. Obtain the Fourier expansion of f(x) = e^{-ax} in the interval $(-\pi, \pi)$. Deduce that

$$\frac{\pi}{\sinh \pi} = 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Here, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi} = \frac{e^{a\pi} - e^{-a\pi}}{a\pi} = \frac{2\sinh a\pi}{a\pi}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\cos nx + n\sin nx\}\right]_{-\pi}^{\pi} = \frac{2a}{\pi}\left[\frac{(-1)^n \sinh a\pi}{a^2+n^2}\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\sin nx - n\cos nx\}\right]_{-\pi}^{\pi} = \frac{2n}{\pi}\left[\frac{(-1)^n \sinh a\pi}{a^2+n^2}\right]$$

Thus, $f(x) = \dfrac{\sinh a\pi}{a\pi} + \dfrac{2a\sinh a\pi}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx + \frac{2}{\pi}\sinh a\pi\sum_{n=1}^{\infty}\frac{n(-1)^n}{a^2+n^2}\sin nx$

For x = 0, a =1, the series reduces to

$f(0)=1 = \dfrac{\sinh\pi}{\pi} + \dfrac{2\sinh\pi}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$

or $1 = \dfrac{\sinh\pi}{\pi} + \dfrac{2\sinh\pi}{\pi}\left[-\dfrac{1}{2} + \displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}\right]$

or $1 = \dfrac{2\sinh\pi}{\pi}\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$

Thus, $\dfrac{\pi}{\sinh\pi} = 2\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$. This is the desired deduction.

4. Obtain the Fourier expansion of $f(x) = x^2$ over the interval (-π, π). Deduce that

a) $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + ...+\infty$ b) $\dfrac{\pi^2}{6} = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + ...+\infty$

c) $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + ...+\infty$

The function f(x) is even. Hence

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx,$ since $f(x)\cos nx$ is even

$= \dfrac{2}{\pi}\displaystyle\int_0^{\pi} x^2\cos nx dx$,Integrating by parts, we get

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi} = \frac{4(-1)^n}{n^2}$$

Also, $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx dx = 0$, since $f(x)\sin nx$ is odd.

Thus

$$f(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2} \text{ ------------ (i)}$$

a) By putting $x = 0$, in (i) we get,

$$0 = f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos 0}{n^2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$i.e. -\frac{\pi^2}{3} = 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots\ldots\ldots\ldots\right)$$

$$\text{i.e. } \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots\ldots$$

b) by putting $x = \pi$, we get,

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \Rightarrow \sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\text{Hence, } \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots$$

c) by adding (a) and (b) we get,

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots\ldots + 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots$$

$$i.e. \frac{3\pi^2}{12} = 2 + \frac{2}{3^2} + \frac{2}{5^2} + \ldots\ldots\ldots \Rightarrow \frac{\pi^2}{4} = 2\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots\ldots\right)$$

$$\text{Or } \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots\ldots$$

Functions having points of discontinuity:

5. Obtain the Fourier expansion of

$$f(x) = \begin{cases} x, 0 \le x \le \pi \\ 2\pi - x, \pi \le x \le 2\pi \end{cases}$$

Deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$

Here, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)dx + \int_{\pi}^{2\pi} f(x)dx\right\} = \frac{1}{\pi}\left\{\int_0^{\pi} xdx + \int_{\pi}^{2\pi} (2\pi - x)dx\right\}$

$$= \frac{1}{\pi}\left\{\frac{x^2}{2}\Bigg|_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)\Bigg|_{\pi}^{2\pi}\right\} = \frac{1}{\pi}\left\{\frac{\pi^2}{2} + 4\pi^2 - \frac{4\pi^2}{2} - (2\pi^2 - \frac{\pi^2}{2})\right\}$$

$$= \frac{1}{2\pi}\left\{\pi^2 + 8\pi^2 - 4\pi^2 - 4\pi^2 + \pi^2\right\} = \pi \Rightarrow \frac{a_0}{2} = \frac{\pi}{2}.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\cos nx dx + \int_{\pi}^{2\pi} f(x)\cos nx dx\right\}$$

$$=\frac{1}{\pi}\left\{\int_0^{\pi} x\cos nx dx+\int_{\pi}^{2\pi}(2\pi-x)\cos nx dx\right\}$$, integrating using Bernoulli's generalized rule

$$=\frac{1}{\pi}\left\{\left[x\left(\frac{\sin nx}{n}\right)-(1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}+\left[(2\pi-x)\left(\frac{\sin nx}{n}\right)-(-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$=\frac{1}{\pi}\left\{\left[\frac{x\sin nx}{n}+\frac{\cos nx}{n^2}\right]_0^{\pi}+\left[(2\pi-x)\left(\frac{\sin nx}{n}\right)-\frac{\cos nx}{n^2}\right]_{\pi}^{2\pi}\right\}$$

$$=\frac{1}{\pi}\left\{\left(\frac{(-1)^n}{n^2}-\frac{1}{n^2}\right)+\left(0-\frac{1}{n^2}-\left(0-\frac{(-1)^n}{n^2}\right)\right)\right\}$$

$$=\frac{1}{\pi}\left\{\frac{2(-1)^n-2}{n^2}\right\}=\frac{2}{\pi n^2}\left\{(-1)^n-1\right\}$$

Also, $b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx=\frac{1}{\pi}\left\{\int_0^{\pi} f(x)\sin nx dx+\int_{\pi}^{2\pi} f(x)\sin nx dx\right\}$,

$$=\frac{1}{\pi}\left\{\int_0^{\pi} x\sin nx dx+\int_{\pi}^{2\pi}(2\pi-x)\sin nx dx\right\}$$, integrating using Bernoulli's generalized rule

$$=\frac{1}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right)-(1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}+\left[(2\pi-x)\left(\frac{-\cos nx}{n}\right)-(-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$=\frac{1}{\pi}\left\{\left[-\frac{x\cos nx}{n}+\frac{\sin nx}{n^2}\right]_0^{\pi}+\left[-(2\pi-x)\frac{\cos nx}{n}-\frac{\sin nx}{n^2}\right]_{\pi}^{2\pi}\right\}$$

$$=\frac{1}{\pi}\left\{\frac{-\pi(-1)^n}{n}+0-\left(\frac{-\pi(-1)^n}{n}-0\right)\right\}=\frac{1}{\pi}\left\{\frac{-\pi(-1)^n}{n}+\frac{\pi(-1)^n}{n}\right\}=0$$

Thus the Fourier series of f(x) is $f(x)=\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n-1\right]\cos nx$

For x = π , we get

$$f(\pi)=\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n-1\right]\cos n\pi$$

But $(-1)^n-1=0$, if n is even & $(-1)^n-1=-2$, if n is odd.

(since f(x) is discontinuous at $x=\pi$, $f(\pi)=\frac{1}{2}\left[f(\pi-0)+f(\pi+0)\right]=\frac{1}{2}(x+2\pi-x)=\pi$)

$$\therefore\ \pi=\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-2\cos(2n-1)\pi}{(2n-1)^2}$$

Thus, $\frac{\pi}{2}=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-2(-1)}{(2n-1)^2}\Rightarrow\frac{\pi^2}{4}=\sum_{n=1}^{\infty}\frac{2}{(2n-1)^2}\Rightarrow\frac{\pi^2}{8}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots$

This is the series as required.

6. Obtain the Fourier expansion of

$$f(x) = \begin{cases} -\pi, -\pi < x < 0 \\ x, 0 < x < \pi \end{cases} \quad \text{hence deduce that } \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

Here, $a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi dx + \int_{0}^{\pi} x dx\right] = \frac{1}{\pi}\left\{[-\pi x]\Big|_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi}\right\} = \frac{1}{\pi}\left[-\pi(0-(-\pi)) + \frac{1}{2}(\pi^2 - 0)\right]$

$$= \frac{1}{\pi}\left(-\pi^2 - \frac{\pi^2}{2}\right) = -\frac{1}{\pi} \times \frac{\pi^2}{2} = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx dx + \int_{0}^{\pi} x \cos nx dx\right]$$

$$= \frac{1}{\pi}\left[\left(-\pi \frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(x \frac{\sin nx}{n} - (1)\frac{-\cos nx}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + 0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx dx + \int_{0}^{\pi} x \sin nx dx\right] =$$

$$= \frac{1}{\pi}\left[\left(-\pi \frac{-\cos nx}{n}\right)_{-\pi}^{0} + \left\{x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(\cos 0 - (-1)^n) - \frac{\pi}{n}(-1)^n + 0\right] = \frac{1}{\pi}\left[\frac{\pi}{n}(1 - 2(-1)^n\right] = \frac{1}{n}\left[1 - 2(-1)^n\right]$$

$\therefore$ Fourier series is

$$f(x) = \frac{-\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos nx + \sum_{n=1}^{\infty}\left[\frac{1 - 2(-1)^n}{n}\right]\sin nx$$

Note that the point x = 0 is a point of discontinuity of $f(x)$. Here $f(x^+) = 0$, $f(x^-) = -\pi$ at x = 0. Hence $\frac{1}{2}[f(x^+) + f(x^-)] = \frac{1}{2}(0 - \pi) = \frac{-\pi}{2}$

The Fourier expansion of f(x) at x = 0 becomes

$$\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]$$

$$or\ \frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]$$

But $(-1)^n - 1 = 0$, if n is even & $(-1)^n - 1 = -2$, if n is odd.

$$\therefore \frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{2}{(2n-1)^2} \Rightarrow \frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots$$

Hence $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + ...$

7. Obtain the Fourier series of $f(x) = 1 - x^2$ over the interval (-1, 1).

The given function is even as f(-x) = f(x). Also period of f(x) is 1-(-1) = 2 $\Rightarrow 2l = 2 \Rightarrow l = 1$

Here $a_0 = \frac{1}{1}\int_{-1}^{1} f(x)dx = 2\int_{0}^{1} f(x)dx = 2\int_{0}^{1}(1 - x^2)dx = 2\left[x - \frac{x^3}{3}\right]_0^1 = \frac{4}{3}$ as f(x) is even.

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)dx = 2\int_{0}^{1} f(x)\cos(n\pi x)dx$$

$= 2\int_{0}^{1}(1 - x^2)\ \cos(n\pi x)dx$ [as $f(x)\cos n\pi x$ is even. Integrating by parts, we get

$$= 2\left[(1 - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(\frac{-\cos n\pi x}{(n\pi)^2}\right) + (-2)\left(\frac{-\sin n\pi x}{(n\pi)^3}\right)\right]_0^1$$

$$= 2\left[[0 - 0 - \frac{2}{(n\pi)^2}[(-1)^n - 0] + \frac{2}{(n\pi)^3}(0)\right] = \frac{4(-1)^{n+1}}{n^2\pi^2}$$

$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)dx = 0$, since $f(x)\sin(n\pi x)$ is odd.

The Fourier series of f(x) is f (x) = $\frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\pi x)$

8. Obtain the Fourier expansion of

$$f(x) = \begin{cases} 1 + \frac{4x}{3}\ in - \frac{3}{2} < x \le 0 \\ 1 - \frac{4x}{3}\ in 0 \le x < \frac{3}{2} \end{cases}$$ .Hence deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + ...$

The period of f(x) is $\frac{3}{2} - \left(\frac{-3}{2}\right) = 3 \Rightarrow 2l = 3 \Rightarrow l = 3/2$

Also f(-x) = f(x). Hence f(x) is even

$$a_0 = \frac{1}{3/2}\int_{-3/2}^{3/2} f(x)dx = \frac{2}{3/2}\int_{0}^{3/2} f(x)dx = \frac{4}{3}\int_{0}^{3/2}\left(1 - \frac{4x}{3}\right)dx = 0$$

$$= \frac{4}{9}\int_{0}^{3/2}(3 - 4x)dx = \frac{4}{9}\left[3x - \frac{4x^2}{2}\right]_0^{3/2} = \frac{4}{9}\left[3x - 2x^2\right]_0^{3/2} = \frac{4}{9}\left[\frac{9}{2} - 2\times\frac{9}{4}\right] = 0$$

$$a_n = \frac{1}{3/2}\int_{-3/2}^{3/2} f(x)\cos\left(\frac{n\pi x}{3/2}\right)dx = \frac{2}{3/2}\int_{0}^{3/2} f(x)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{4}{3}\int_{0}^{3/2}\left(1 - \frac{4x}{3}\right)cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{4}{3}\left[\left(1-\frac{4x}{3}\right)\left(\frac{sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right)-\left(\frac{-4}{3}\right)\left(\frac{-cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right)\right]_0^{3/2}$$

$$= \frac{4}{3}\left[0-\frac{4}{3}\times\frac{9}{4n^2\pi^2}((-1)^n-1)\right] = \frac{4}{n^2\pi^2}\left[1-(-1)^n\right] = \frac{8}{n^2\pi^2}\text{, n = 1, 3, 5, ………}$$

Also, $b_n = \frac{1}{3}\int_{-3/2}^{3/2} f(x) sin\left(\frac{n\pi x}{3/2}\right)dx = 0$, as $f(x) sin\left(\frac{n\pi x}{3/2}\right)$ is an odd function.

Thus f(x) = $\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}cos\left(\frac{2n\pi x}{3}\right), n = 1,3,5,........$

putting x=0, we get f(0) = $\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$

or 1 = $\frac{8}{\pi^2}\left[1+\frac{1}{3^2}+\frac{1}{5^2}+......\right]$

Thus, $\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+......$

Exercises: 1) Obtain the Fourier series of $f(x) = \frac{\pi - x}{2}$ in $0 < x < 2\pi$. Hence deduce that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+.........= \frac{\pi}{4}$

2) If $f(x) = x(2\pi - x)$ in $0 \le x \le 2\pi$ show that

$f(x) = \frac{2\pi^2}{3} - 4\left(\frac{cos x}{1^2}+\frac{cos 2x}{2^2}+\frac{cos 3x}{3^2}+..........\right)$ hence deduce that

$\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+......$

3) Find a Fourier series in $(-\pi, \pi)$ to represent a) $f(x) = x - x^2$ b) $f(x) = x + x^2$
Hence deduce that

a) $\frac{\pi^2}{12} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+......+\infty$

b) $\frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+......+\infty$

c) $\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+......+\infty$

4) Obtain the Fourier series for the function $f(x) = \begin{cases} 0 & in \ -\pi \le x \le 0 \\ \sin x & in \ \ 0 \le x \le \pi \end{cases}$

Hence deduce that

1. $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+.......=\frac{1}{2}$ 2. $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}+.......=\frac{\pi-2}{4}$

5) If $f(x)=\left(\frac{\pi-x}{2}\right)^2$ in the range 0 to 2π, show that $f(x)=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$

6) An alternating current after passing through a rectifier has the form

$$i=\begin{cases} I_0 \sin x & for \quad 0\le x\le \pi \\ 0 & for \quad \pi \le x \le 2\pi \end{cases}$$

Where I_0 is the maximum current and the period is 2π. Express i as a Fourier series.

7) Obtain the Fourier series for the function

$$f(x)=\begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 < x < \pi \end{cases} \qquad \text{Where } f(x+2\pi)=f(x)$$

8) Obtain the Fourier series for the function

$$f(x)=\begin{cases} x^2, & 0\le x\le \pi \\ -x^2, & -\pi \le x \le 0 \end{cases}$$

9) Expand $f(x)=e^{-x}$ as a Fourier series in the interval $(-l,l)$

10) If $f(x)=\begin{cases} \pi x, & 0\le x\le 1 \\ \pi(2-x), & 1\le x\le 2 \end{cases}$; show that in the interval $(0,2)$

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$

Answers: 1) $f(x)=\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$

3) a) $x-x^2=\frac{-\pi^2}{3}+4\sum_{1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx+2\sum_{1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$

b) $x-x^2=\frac{\pi^2}{3}+4\sum_{1}^{\infty}\frac{(-1)^{n}}{n^2}\cos nx+2\sum_{1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$

4) $f(x)=\frac{1}{\pi}+\sum_{2}^{\infty}\frac{-1}{\pi(n^2-1)}\{1+(-1)^n\}\cos nx+\frac{1}{2}\sin x.$

6) $i=\frac{I_0}{\pi}+\frac{1}{2}I_0\sin x-\frac{2I_0}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$

7) $f(x)=\frac{\pi^2}{6}-2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos nx}{n^2}-\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{2}{n^3}-\frac{\pi^2}{n}\right)\sin nx$

8) $f(x) = 2\left(\pi - \frac{4}{\pi}\right)\sin x - \pi \sin 2x + \frac{2}{3}\left(\pi - \frac{4}{9\pi}\right)\sin 3x - \frac{\pi}{2}\sin 4x + ...$

9) $e^{-x} = \sinh l \left\{ \frac{1}{l} - 2l \sum_{n=1}^{\infty} \frac{1}{l^2 + n^2\pi^2} \cos\frac{n\pi x}{l} - 2\pi \sum \frac{n}{l^2 + n^2\pi^2} \sin\frac{n\pi x}{l} \right\}$

HALF-RANGE FOURIER SERIES

The Fourier expansion of the periodic function f(x) of period 2*l* may contain both sine and cosine terms. Many a time it is required to obtain the Fourier expansion of f(x) in the interval (0, *l*) which is regarded as half interval. The definition can be extended to the other half in such a manner that the function becomes even or odd. This will result in cosine series or sine series only.

Sine series :

Suppose f(x) = φ(x) is given in the interval (0, *l*). Then we define f(x) = -φ(-x) in (-*l*, 0). Hence f(x) becomes an odd function in (-*l* , *l*). The Fourier series then is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad (11)$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

The series (11) is called half-range sine series over (0, *l*).

Putting l=π in (11), we obtain the half-range sine series of f(x) over (0, π) given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

Cosine series :

Let us define

$$f(x) = \begin{cases} \phi(x) \quad in \quad (0, l).........given \\ \phi(-x) \quad in \quad (-l, 0)........in\ order\ to\ make\ the\ function\ even \end{cases}$$

Then the Fourier series of f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \qquad (12)$$

where,

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

The series (12) is called half-range cosine series over (0,*l*)

Putting l = π in (12), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx \quad \text{n} = 1,2,3,\ldots..$$

Examples :

1. Expand f(x) = x(π-x) as half-range sine series over the interval (0,π).
We have,

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx dx$$

$$= \frac{2}{\pi}\int_0^\pi (\pi x - x^2)\sin nx dx$$

Integrating by parts, we get

$$b_n = \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{4}{n^3\pi}\left[1 - (-1)^n\right]$$

The sine series of f(x) is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^3}\left[1 - (-1)^n\right]\sin nx$$

2. Obtain the cosine series of

$$f(x) = \begin{cases} x, 0 < x < \frac{\pi}{2} \\ \pi - x, \frac{\pi}{2} < x < \pi \end{cases} \qquad over(0,\pi)$$

Here

$$a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right]$$

Performing integration by parts and simplifying, we get

$$a_n = -\frac{2}{n^2\pi}\left[1 + (-1)^n - 2\cos\left(\frac{n\pi}{2}\right)\right]$$

$$= -\frac{8}{n^2\pi}, n = 2,6,10,.....$$

Thus, the Fourier cosine series is

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + ...\infty\right]$$

3. Obtain the half-range cosine series of f(x) = c-x in $0<x<c$

Here $a_0 = \frac{2}{c}\int_0^c (c-x)\,dx = c$, $a_n = \frac{2}{c}\int_0^c (c-x)\cos\left(\frac{n\pi x}{c}\right)dx$

Integrating by parts and simplifying we get, $a_n = \frac{2c}{n^2\pi^2}\left[1-(-1)^n\right]$

The cosine series is given by

$$f(x) = \frac{c}{2} + \frac{2c}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi x}{c}\right)$$

Exercices:

I. Obtain the half-range sine series of the following functions over the specified intervals:

1. f(x) = cosx over $(0,\pi)$ 2. f(x) = $\sin^3 x$ over $(0,\pi)$ 3. f(x) = $lx - x^2$ over $(0, l)$

II. Obtain the half-range cosine series of the following functions over the specified intervals:

1. f(x) = x^2 over $(0,\pi)$ 2. f(x) = xsinx over $(0,\pi)$ 3. f(x) = $(x-1)^2$ over (0, 1)

4. $f(x) = \begin{cases} kx, 0 \le x \le \frac{l}{2} \\ k(l-x), \frac{l}{2} \le x \le l \end{cases}$

HARMONIC ANALYSIS

The Fourier series of a **known** function f(x) in a given interval may be found by finding the Fourier coefficients. The method described cannot be employed when f(x) is not known explicitly, but defined through the values of the function at some equidistant points. In such a case, the integrals in Euler's formulae cannot be evaluated. Harmonic analysis is the process of finding the Fourier coefficients numerically.

To derive the relevant formulae for Fourier coefficients in Harmonic analysis, we employ the following result:

The mean value of a continuous function f(x) over the interval (a, b) denoted by $[f(x)]$ is defined as $\quad [f(x)] = \frac{1}{b-a}\int_a^b f(x)dx$.

The Fourier coefficients defined through Euler's formulae, (1), (2), (3) may be redefined as

$$a_0 = 2\left[\frac{1}{2l}\int_a^{a+2l} f(x)dx\right] = 2[f(x)]$$

$$a_n = 2\left[\frac{1}{2l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx\right] = 2\left[f(x)\cos\left(\frac{n\pi x}{l}\right)\right]$$

$$b_n = 2\left[\frac{1}{2l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right] = 2\left[f(x)\sin\left(\frac{n\pi x}{l}\right)\right]$$

Using these in (5), we obtain the Fourier series of f(x). The term $a_1\cos x + b_1\sin x$ is called the first harmonic or fundamental harmonic, the term $a_2\cos 2x + b_2\sin 2x$ is called the second harmonic and so on. The amplitude of the first harmonic is $\sqrt{a_1^2 + b_1^2}$ and that of second harmonic is $\sqrt{a_2^2 + b_2^2}$ and so on.

Examples

1. Find the first two harmonics of the Fourier series of f(x) given the following table:

X	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
f(x)	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Note that the values of y = f(x) are spread over the interval 0≤ x ≤ 2π and f(0) = f(2π) = 1.0. Hence the function is periodic and so we omit the last value f(2π) = 0. We prepare the following table to compute the first two harmonics.

x^o	$y = f(x)$	$\cos x$	$\cos 2x$	$\sin x$	$\sin 2x$	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
0^o	1.0	1	1	0	0	1	1	0	0
60^o	1.4	0.5	-0.5	0.866	0.866	0.7	-0.7	1.2124	1.2124

120^o	1.9	-0.5	-0.5	0.866	-0.866	-0.95	-0.95	1.6454	-1.6454
180^o	1.7	-1	1	0	0	-1.7	1.7	0	0
240^o	1.5	-0.5	-0.5	-0.866	0.866	-0.75	-0.75	-1.299	1.299
300^o	1.2	0.5	-0.5	-0.866	-0.866	0.6	-0.6	-1.0392	-1.0392
Total						-1.1	-0.3	0.519	-0.1732

We have $a_n = 2\left[f(x)\cos\left(\frac{n\pi x}{l}\right)\right] = 2[y\cos nx]$ $\quad b_n = 2\left[f(x)\sin\left(\frac{n\pi x}{l}\right)\right] = 2[y\sin nx]$

as the length of interval = $2l = 2\pi$ or $l = \pi$

Putting, n =1, 2, we get

$$a_1 = 2[y\cos x] = \frac{2\sum y\cos x}{6} = \frac{2(-1.1)}{6} == -0.367$$

$$a_2 = 2[y\cos 2x] = \frac{2\sum y\cos 2x}{6} = \frac{2(-0.3)}{6} = -0.1$$

$$b_1 = 2[y\sin x] = \frac{2\sum y\sin x}{6} = 0.173$$

$$b_2 = 2[y\sin 2x] = \frac{2\sum y\sin 2x}{6} = -0.0577$$

The first two harmonics are $a_1\cos x + b_1\sin x$ and $a_2\cos 2x + b_2\sin 2x$ .That is

$(-0.367\cos x + 1.0392\sin x)$ and $(-0.1\cos 2x - 0.0577\sin 2x)$

2. Express y as a Fourier series up to the third harmonic given the following values:

X	0	1	2	3	4	5
Y	4	8	15	7	6	2

The values of y at x = 0, 1, 2, 3, 4, 5 are given and hence the interval of x should be

$0 \le x < 6$. The length of the interval = 6 - 0 = 6, so that $2l = 6$ or $l = 3$.

The Fourier series up to the third harmonic is

$$y = \frac{a_0}{2} + \left(a_1\cos\frac{\pi x}{l} + b_1\sin\frac{\pi x}{l}\right) + \left(a_2\cos\frac{2\pi x}{l} + b_2\sin\frac{2\pi x}{l}\right) + \left(a_3\cos\frac{3\pi x}{l} + b_3\sin\frac{3\pi x}{l}\right) \text{ or}$$

$$y = \frac{a_0}{2} + \left(a_1\cos\frac{\pi x}{3} + b_1\sin\frac{\pi x}{3}\right) + \left(a_2\cos\frac{2\pi x}{3} + b_2\sin\frac{2\pi x}{3}\right) + \left(a_3\cos\frac{3\pi x}{3} + b_3\sin\frac{3\pi x}{3}\right)$$

Put $\theta = \frac{\pi x}{3}$, then

$$y = \frac{a_0}{2} + (a_1\cos\theta + b_1\sin\theta) + (a_2\cos 2\theta + b_2\sin 2\theta) + (a_3\cos 3\theta + b_3\sin 3\theta) \qquad (1)$$

We prepare the following table using the given values:

x	$\theta = \frac{\pi x}{3}$	y	$y\cos\theta$	$y\cos 2\theta$	$y\cos 3\theta$	$y\sin\theta$	$y\sin 2\theta$	$y\sin 3\theta$

0	0	04	4	4	4	0	0	0
1	60^0	08	4	-4	-8	6.928	6.928	0
2	120^0	15	-7.5	-7.5	15	12.99	-12.99	0
3	180^0	07	-7	7	-7	0	0	0
4	240^0	06	-3	-3	6	-5.196	5.196	0
5	300^0	02	1	-1	-2	-1.732	-1.732	0
Total		42	-8.5	-4.5	8	12.99	-2.598	0

$a_0 = 2[f(x)] = 2[y] = \frac{2\sum y}{6} = \frac{1}{3}(42) = 14$, $a_1 = 2[y\cos\theta] = \frac{2}{6}(-8.5) = -2.833,$

$b_1 = 2[y\sin\theta] = \frac{2}{6}(12.99) = 4.33,$ $a_2 = 2[y\cos 2\theta] = \frac{2}{6}(-4.5) = -1.5,$

$b_2 = 2[y\sin 2\theta] = \frac{2}{6}(-2.598) = -0.866,$ $a_3 = 2[y\cos 3\theta] = \frac{2}{6}(8) = 2.667,$ $b_3 = 2[y\sin 3\theta] = 0$

Using these in (1), we get

$$y = 7 - 2{,}833\cos\left(\frac{\pi x}{3}\right) + (4.33)\sin\left(\frac{\pi x}{3}\right) - 1.5\cos\left(\frac{2\pi x}{3}\right) - 0.866\sin\left(\frac{2\pi x}{3}\right) + 2.667\cos\pi x$$

This is the required Fourier series up to the third harmonic.

3. The following table gives the variations of a periodic current A over a period T:

T (secs)	0	T/6	T/3	T/2	2T/3	5T/6	T
A (amp)	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Show that there is a constant part of 0.75amp. in the current A and obtain the amplitude of the first harmonic.

Note that the values of A at t=0 and t =T are the same. Hence A (t) is a periodic function of period T. Let us denote $\theta = \left(\frac{2\pi}{T}\right)t$. We have

$$a_0 = 2[A], \quad a_1 = 2\left[A\cos\left(\frac{2\pi}{T}\right)t\right] = 2[A\cos\theta], \quad b_1 = 2\left[A\sin\left(\frac{2\pi}{T}\right)t\right] = 2[A\sin\theta] \text{------ (1)}$$

We prepare the following table:

t	$\theta = \frac{2\pi t}{T}$	A	cosθ	sinθ	A cosθ	A sinθ
0	0	1.98	1	0	1.98	0
T/6	60^0	1.30	0.5	0.866	0.65	1.1258
T/3	120^0	1.05	-0.5	0.866	-0.525	0.9093

T/2	180^0	1.30	-1	0	-1.30	0
2T/3	240^0	-0.88	-0.5	-0.866	0.44	0.7621
5T/6	300^0	-0.25	0.5	-0.866	-0.125	0.2165
Total		4.5			1.12	3.0137

Using the values of the table in (1), we get

$$a_0 = \frac{2\sum A}{6} = \frac{4.5}{3} = 1.5,\ a_1 = \frac{2\sum A\cos\theta}{6} = \frac{1.12}{3} = 0.3733,\quad b_1 = \frac{2\sum A\sin\theta}{6} = \frac{3.0137}{3} = 1.0046$$

The Fourier expansion up to the first harmonic is

$$A = \frac{a_0}{2} + a_1\cos\left(\frac{2\pi t}{T}\right) + b_1\sin\left(\frac{2\pi t}{T}\right) == 0.75 + 0.3733\cos\left(\frac{2\pi t}{T}\right) + 1.0046\sin\left(\frac{2\pi t}{T}\right)$$

The expression shows that A has a constant part 0.75 in it. Also the amplitude of the first harmonic is $\sqrt{a_1^2 + b_1^2}$ = 1.0717.

ASSIGNMENT:

1. The displacement y of a part of a mechanism is tabulated with corresponding angular movement x^0 of the crank. Express y as a Fourier series up to the third harmonic.

x^0	0	30	60	90	120	150	180	210	240	270	300	330
y	1.80	1.10	0.30	0.16	1.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

2. Obtain the Fourier series of y up to the second harmonic using the following table:

x^0	45	90	135	180	225	270	315	360
y	4.0	3.8	2.4	2.0	-1.5	0	2.8	3.4

3. Obtain the constant term and the coefficients of the first sine and cosine terms in the Fourier expansion of y as given in the following table:

x	0	1	2	3	4	5
y	9	18	24	28	26	20

4. Find the Fourier series of y up to the second harmonic from the following table:

x	0	2	4	6	8	10	12
Y	9.0	18.2	24.4	27.8	27.5	22.0	9.0

5. Obtain the first three coefficients in the Fourier cosine series for y, where y is given in the following table:

x	0	1	2	3	4	5
y	4	8	15	7	6	2

6. The turning moment T is given for a series of values of the crank angle $\theta^0 = 75^0$.

θ^0	0	30	60	90	120	150	180
T	0	5224	8097	7850	5499	2626	0

Obtain the first four terms in a series of sines to represent T and calculate T at $\theta = 75^0$.

Module-4 – Fourier Transforms & Z-Transforms

- **Infinite Fourier transforms:** Definition, Fourier sine, and cosine transform. Inverse Fourier transforms Inverse Fourier cosine and sine transforms. Problems.
- **Z-transforms:** Definition, Standard z-transforms, Damping, and shifting rules, Problems. Inverse z-transform and applications to solve difference equations
- **Self-study:** Convolution theorems of Fourier and z-transforms

FOURIER TRANSFORMS

Introduction

Fourier Transform is a technique employed to solve ODE's, PDE's, IVP's, BVP's and Integral equations.

Infinite Fourier Transform

Let f(x) be a real valued, differentiable function that satisfies the following conditions:

1) $f(x)$ and its derivative $f'(x)$ are continuous, or have only a finite number of simple discontinuities in every finite interval, and

2) The integral $\int_{-\infty}^{\infty} |f(x)|\, dx$ exists.

Also, let s be non – zero real parameter. Then infinite Fourier Transform of $f(x)$ denoted by $\hat{f}(s)$ or $F[f(x)]$ or F(s) is defined by

$$\hat{f}(s) = F(s) = F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{isx}dx$$, provided the integral exists.

The infinite Fourier Transform is also called complex Fourier Transform or just the Fourier Transform. The Inverse Fourier Transform of $\hat{f}(s)$ denoted by $F^{-1}\left[\hat{f}(s)\right]$ is defined by

$$F^{-1}\left[\hat{f}(s)\right] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(s)e^{-isx}ds$$

Note: The function f(x) is said to be self reciprocal with respect to Fourier transform if $\hat{f}(s) = f(s)$.

Basic Properties:

Below we prove some basic properties of Fourier Transforms:

1. **Linearity Property**

For any two functions f(x) and ϕ(x) (whose Fourier Transforms exist) and any two constants a and b, $F[af(x)+b\phi(x)] = aF[f(x)]+bF[\phi(x)]$

Proof: By definition, we have

$$F[af(x)+b\phi(x)] = \int_{-\infty}^{\infty}[af(x)+b\phi(x)]e^{isx}dx = a\int_{-\infty}^{\infty} f(x)e^{isx}dx + b\int_{-\infty}^{\infty}\phi(x)e^{isx}dx$$

$= aF[f(x)]+bF[\phi(x)]$. This is the desired property.

In particular, if a = b = 1, we get $F[f(x)+\phi(x)] = F[f(x)]+F[\phi(x)]$

Again if a = -b = 1, we get $F[f(x)-\phi(x)] = F[f(x)]-F[\phi(x)]$

2. **Change of Scale Property**

If $\hat{f}(s) = F[f(x)]$, then for any non – zero constant a, we have $F[f(ax)] = \frac{1}{a}\hat{f}\left(\frac{s}{a}\right),\ a \neq 0$

Proof: By definition, we have $F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$\therefore F[f(ax)] = \int_{-\infty}^{\infty} f(ax)e^{isx}dx \qquad \text{put } ax = t \Rightarrow adx = dt \Rightarrow dx = \frac{dt}{a}$$

$$= \int_{-\infty}^{\infty} f(t)e^{is\left(\frac{t}{a}\right)}\frac{dt}{a} = \frac{1}{a}\int_{-\infty}^{\infty} f(t)e^{i(s/a)t}dt = \frac{1}{a}\hat{f}\left(\frac{s}{a}\right)$$

3. **Shifting Properties:**

For any real constant 'a', $(i)\quad F[f(x-a)] = e^{isa}\hat{f}(s) \quad (ii)\ F[e^{iax}f(x)] = \hat{f}(s+a)$, Where $\hat{f}(s) = F[f(x)]$

Proof : (i) We have $F[f(x)] = \hat{f}(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx$

Hence, $F[f(x-a)] = \int_{-\infty}^{\infty} f(x-a)e^{isx}dx$, set $x - a = t \Rightarrow dx = dt$, then

$$F[f(x-a)] = \int_{-\infty}^{\infty} f(t)e^{is(t+a)}dt = e^{ias}\int_{-\infty}^{\infty} f(t)e^{ist}dt = e^{isa}\hat{f}(s)$$

ii) We have $\hat{f}(s+a) = \int_{-\infty}^{\infty} f(x)e^{i(s+a)x}dx = \int_{-\infty}^{\infty}\left[f(x)e^{iax}\right]e^{isx}dx$

$$= \int_{-\infty}^{\infty} g(x)e^{isx}dx, \text{ where } g(x) = f(x)e^{iax}$$

$= F[g(x)] = F[e^{iax}f(x)]$ This is the desired result.

4. **Modulation Property:** *If* $F[f(x)] = \hat{f}(s)$, then $F[f(x)\cos ax] = \frac{1}{2}\left[\hat{f}(s+a) + \hat{f}(s-a)\right]$

where 'a' is a real constant.

Proof: We have $\cos ax = \dfrac{e^{iax} + e^{-iax}}{2}$

Hence $F[f(x)\cos ax] = F\left[f(x)\left(\dfrac{e^{iax} + e^{-iax}}{2}\right)\right]$

$$= \frac{1}{2}F\left[f(x)e^{iax} + f(x)e^{-iax}\right] = \frac{1}{2}\left[F[f(x)e^{iax}] + F[f(x)e^{-iax}]\right]$$

$= \frac{1}{2}\left[\hat{f}(s+a) + \hat{f}(s-a)\right]$, by using linearity and shift properties.

This is desired property.

Note: Similarly $F\left[f(x)\sin ax\right]=\frac{1}{2}\left[\hat{f}(s+a)-\hat{f}(s-a)\right]$

Examples

1. Find the Fourier transform of $f(x)=\begin{cases}1 \; for|x|<1\\ 0 \; for|x|>1\end{cases}$ hence evaluate $\int_0^\infty \frac{\sin x}{x}dx$

<< The Fourier transform of f(x) is given by

$$\hat{f}(s)=\int_{-\infty}^{\infty} f(x)e^{isx}dx=\int_{-\infty}^{-1}0.e^{isx}dx+\int_{-1}^{1}1.e^{isx}dx+\int_{1}^{\infty}0.e^{isx}dx=\frac{e^{isx}}{is}\Big|_{-1}^{1}=\frac{e^{is}-e^{-is}}{is}=\frac{2\sin s}{s}, s\neq 0$$

& for $s=0,\ \hat{f}(s)=2$.

Now by the inverse Fourier transform we have

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(s)e^{-isx}ds \text{ Or } \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin s}{s}e^{-isx}ds=\begin{cases}1 \; for|x|<1\\ 0 \; for|x|>1\end{cases}$$

By putting x = 0 we get,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin s}{s}ds=1\Rightarrow \int_{-\infty}^{\infty}\frac{\sin s}{s}ds=\pi\Rightarrow 2\int_0^\infty\frac{\sin s}{s}ds=\pi$$, as the integrand is even

$$\Rightarrow \int_0^\infty \frac{\sin s}{s}ds=\frac{\pi}{2} \text{ 0r } \int_0^\infty\frac{\sin x}{x}dx=\frac{\pi}{2}$$

2. Find the Fourier transform of $f(x)=\begin{cases}1-x^2 \; for|x|\le 1\\ 0 \quad for|x|>1\end{cases}$ Hence evaluate

$$\int_0^\infty \frac{x\cos x-\sin x}{x^3}\cos\frac{x}{2}dx$$

<< The Fourier transform of f(x) is given by

$$\hat{f}(s)=\int_{-\infty}^{\infty}f(x)e^{isx}dx=\int_{-\infty}^{-1}0.e^{isx}dx+\int_{-1}^{1}(1-x^2).e^{isx}dx+\int_1^\infty 0.e^{isx}dx$$

$$=(1-x^2)\frac{e^{isx}}{is}\Big|_{-1}^{1}-(-2x)\frac{e^{isx}}{(is)^2}\Big|_{-1}^{1}+(-2)\frac{e^{isx}}{(is)^3}\Big|_{-1}^{1}$$

$$=0+\frac{2}{-s^2}(e^{is}+e^{-is})-\frac{2}{-is^3}(e^{is}-e^{-is})=\frac{-4\cos s}{s^2}+\frac{4\sin s}{s^3}=-\frac{4}{s^3}(s\cos s-\sin s)$$

Now by inversion formula, we have

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(s)e^{-isx}ds$$

$$\text{Or } \frac{1}{2\pi}\int_{-\infty}^{\infty}-\frac{4}{s^3}(s\cos s-\sin s)e^{-isx}ds=\begin{cases}1-x^2, |x|\le 1\\ 0, \quad |x|>1\end{cases}$$

Putting $x=1/2$, we get

$$-\frac{4}{2\pi}\int_{-\infty}^{\infty}\frac{1}{s^3}(s\cos s-\sin s)e^{-is/2}ds=\frac{3}{4}$$

Or $\displaystyle\int_{-\infty}^{\infty}\frac{s\cos s-\sin s}{s^3}\left(\cos\frac{s}{2}-i\sin\frac{s}{2}\right)ds=-\frac{3\pi}{8}$

Or $\displaystyle\int_{-\infty}^{\infty}\frac{s\cos s-\sin s}{s^3}\cos\frac{s}{2}ds=-\frac{3\pi}{8}\Rightarrow\int_{0}^{\infty}\frac{s\cos s-\sin s}{s^3}\cos\frac{s}{2}ds=-\frac{3\pi}{16}$, since the integrand
is even

$$\int_{0}^{\infty}\frac{x\cos x-\sin x}{x^3}.\cos\frac{x}{2}dx=-\frac{3\pi}{16}$$

3. Find the Fourier Transform of the function $f(x)=e^{-a|x|}$ where $a>0$

The Fourier transform of f(x) is given by

$$\hat{f}(s)=\int_{-\infty}^{\infty}f(x)e^{isx}dx=\int_{-\infty}^{\infty}e^{-a|x|}e^{isx}dx=\left[\int_{-\infty}^{0}e^{-a|x|}e^{isx}dx+\int_{0}^{\infty}e^{-a|x|}e^{isx}dx\right]$$

Using the fact that $|x|=x, 0\le x<\infty$ & $|x|=-x, -\infty<x\le 0$, we get

$$\hat{f}(s)=\left[\int_{-\infty}^{0}e^{ax}e^{isx}dx+\int_{0}^{\infty}e^{-ax}e^{isx}dx\right]=\left[\int_{-\infty}^{0}e^{(a+is)x}dx+\int_{0}^{\infty}e^{-(a-is)x}dx\right]$$

$$=\left[\left\{\frac{e^{(a+is)x}}{(a+is)}\right\}_{-\infty}^{0}+\left\{\frac{e^{-(a-is)x}}{-(a-is)}\right\}_{0}^{\infty}\right]=\left[\frac{1}{(a+is)}+\frac{1}{(a-is)}\right]=\left[\frac{2a}{a^2+s^2}\right]$$

Exercises:

4. Find the Fourier Transform of the function $f(x)=\begin{cases}1, & |x|\le a\\ 0, & |x|>a\end{cases}$

where 'a' is a positive constant. Hence evaluate

$$(i)\quad\int_{-\infty}^{\infty}\frac{\sin s\,a\ \cos s\,x}{s}\,ds\qquad(ii)\quad\int_{0}^{\infty}\frac{\sin s}{s}ds$$

<< For the given function, we have

$$F[f(x)]=\left|\int_{-\infty}^{\infty}f(x)e^{isx}dx\right|$$
$$=\left|\int_{-\infty}^{-a}f(x)e^{isx}dx+\int_{-a}^{a}f(x)e^{isx}dx+\int_{a}^{\infty}f(x)e^{isx}dx\right|$$
$$=\left|\int_{-a}^{a}e^{isx}dx\right|=2\left[\frac{\sin sa}{s}\right]$$
$$Thus\quad F[f(x)]=\hat{f}(s)=2\left(\frac{\sin s\,a}{s}\right)\qquad\text{.........(1)}$$

Inverting $\hat{f}(s)$ by employing inversion formula, we get

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}2\left[\frac{\sin sa}{s}\right]e^{-isx}ds$$
$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin sa(\cos sx-i\sin sx)}{s}ds$$
$$=\frac{1}{\pi}\left[\int_{-\infty}^{\infty}\frac{\sin sa(\cos s\ x)}{s}ds-i\int_{-\infty}^{\infty}\frac{\sin s\ a\sin sx}{s}ds\right]$$

Here, the integrand in the first integral is even and the integrand in the second integral is odd. Hence using the relevant properties of integral here, we get

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin sa\cos sx}{s}ds \qquad \text{or}$$

$$\int_{-\infty}^{\infty}\frac{\sin sa\cos sx}{s}ds=\pi f(x)\ =\begin{cases}\pi, & |x|\le a\\ 0, & |x|>a\end{cases}$$

For x = 0, a = 1, this yields $\int_{-\infty}^{\infty}\frac{\sin s}{s}ds=\pi$

Since the integrand is even, we have $2\int_{0}^{\infty}\frac{\sin s}{s}ds=\pi$

Or $\int_{0}^{\infty}\frac{\sin s}{s}ds=\frac{\pi}{2}$

5. Find the Fourier Transform of $f(x)=e^{-a^2x^2}$ where 'a' is a positive constant.

Deduce that $f(x)=e^{-x^2/2}$ is self reciprocal with respect to Fourier transform.

Here

$$F[f(x)]=\int_{-\infty}^{\infty}e^{-a^2x^2}e^{isx}dx\ =\int_{-\infty}^{\infty}e^{-\left(a^2x^2-isx\right)}dx$$
$$=\int_{-\infty}^{\infty}e^{-\left[\left(ax-\frac{is}{2a}\right)^2+\frac{s^2}{4a^2}\right]}dx$$
$$=e^{-\left(s^2/4a^2\right)}\int_{-\infty}^{\infty}e^{-\left(ax-\frac{is}{2a}\right)^2}dx$$

Setting $t=ax-\frac{is}{2a}$, we get $F[f(x)]=e^{-\left(s^2/4a^2\right)}\int_{-\infty}^{\infty}e^{-t^2}\frac{dt}{a}$

$$=\frac{1}{a}e^{-\left(s^2/4a^2\right)}2\int_{0}^{\infty}e^{-t^2}dt \quad \text{but } \int_0^\infty e^{-t^2}dt=\frac{\sqrt{\pi}}{2}$$

$$=\frac{1}{a}e^{-\left(s^2/4a^2\right)}\sqrt{\pi}, \text{using gamma function.}$$

$$\hat{f}(s) = \frac{\sqrt{\pi}}{a} e^{-\left(s^2/4a^2\right)}$$

This is the desired Fourier Transform of f(x).

For $a^2 = 1/2$ in $f(x) = e^{-a^2x^2}$

we get $f(x) = e^{-x^2/2}$ and hence,

$$\hat{f}(s) = \sqrt{2\pi}\, e^{-s^2/2}$$

Also putting x = s in $f(x) = e^{-x^2/2}$, we get $f(s) = e^{-s^2/2}$.

Hence, $f(s)$ and $\hat{f}(s)$ are same but for constant multiplication by $\sqrt{2\pi}$.

Thus f(s) = $\hat{f}(s)$

It follows that $f(x) = e^{-x^2/2}$ is self reciprocal

6) Find the inverse Fourier Transform of $\hat{f}(s) = e^{-s^2}$

Exercises:

Find the Complex Fourier Transforms of the following functions:

(1) $f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases}$ where 'a' is a positive constant

(2) $f(x) = \begin{cases} 1-|x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

(3) $f(x) = \begin{cases} 0, & x < a \\ 1, & a \le x \le b \\ 0, & x > b \end{cases}$ where 'a' and 'b' are positive constants

(4) $f(x) = \begin{cases} a^2 - x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$

(5) $f(x) = xe^{-a|x|}$ where 'a' is a positive constant

(6) $f(x) = e^{-|x|}$

(7) $f(x) = \cos 2x^2$

(8) $f(x) = \sin 3x^2$

FOURIER SINE TRANSFORMS:

Let f(x) be defined for all positive values of x.

The integral $\int_0^\infty f(x)\sin sxdx$ is called the Fourier sine Transform of $f(x)$. This is denoted by

$\hat{f}_S(s)$ or $F_S[f(x)]$, thus $\hat{f}_S(s) = F_S[f(x)] = \int_0^\infty f(x)\sin sxdx$

The inverse Fourier sine Transform of $\hat{f}_S(s)$ is defined through the integral $\frac{2}{\pi}\int_0^\infty \hat{f}_S(s)\sin sxds$.

This is denoted by f(x) or $F_S^{-1}[f_S(s)]$. *Thus*

$$f(x) = F_S^{-1}[f_S(s)] = \frac{2}{\pi}\int_0^\infty \hat{f}_S(s)\sin sxds$$

Properties

The following are the basic properties of Sine Transforms.

(1) **LINEARITY PROPERTY**

If 'a' and 'b' are two constants, then for two functions f(x) and ϕ(x), we have

$$F_S[af(x)+b\phi(x)] = aF_S[f(x)] + bF_S[g(x)]$$

Proof : By definition, we have

$$F_S[af(x)+b\phi(x)] = \int_0^\infty [af(x)+b\phi(x)]\sin sx\,dx = \int_0^\infty af(x)\sin sxdx + \int_0^\infty b\phi(x)\sin sxdx$$

$$= a\int_0^\infty f(x)\sin sxdx + b\int_0^\infty \phi(x)\sin sxdx = aF_S[f(x)] + bF_S[\phi(x)]$$

This is the desired result. In particular, we have

$$F_S[f(x)+\phi(x)] = F_S[f(x)] + F_S[\phi(x)] \text{ and } F_S[f(x)-\phi(x)] = F_S[f(x)] - F_S[\phi(x)]$$

(2) **CHANGE OF SCALE PROPERTY**

If $F_S[f(x)] = \hat{f}_S(s)$, then for a $\neq$ 0, we have $F_S[f(ax)] = \frac{1}{a}\hat{f}_S\left(\frac{s}{a}\right)$

Proof : We have $F_S[f(ax)] = \int_0^\infty f(ax)\sin s\,x\,dx$

Setting ax = t, we get $F_S[f(ax)] = \int_0^\infty f(t)\sin\left(\frac{s}{a}\right)t\left(\frac{dt}{a}\right)$

$$\frac{1}{a}\int f(t)\sin\left(\frac{s}{a}\right)tdt = \frac{1}{a}\hat{f}_S\left(\frac{s}{a}\right)$$

(3) **MODULATION PROPERTY**

If $F_S[f(x)] = \hat{f}_S(s)$, then for a $\neq$ 0, we have $F_S[f(x)\cos ax] = \frac{1}{2}\left[\hat{f}_S(s+a) + \hat{f}_S(s-a)\right]$

Proof: We have $F_s\left[f(x)\cos ax\right]=\int_0^\infty f(x)\cos ax \sin sx\, dx$

$$=\frac{1}{2}\left[\int_0^\infty f(x)\{\sin(s+a)x+\sin(s-a)x\}dx\right]$$

$$=\frac{1}{2}\left[\int_0^\infty f(x)\sin(s+a)xdx+\int_0^\infty f(x)\sin(s-a)xdx\right]$$

$$=\frac{1}{2}\left[\hat{f}_s(s+a)+\hat{f}_s(s-a)\right],\text{ by using Linearity property.}$$

FOURIER COSINE TRANSFORMS:

Let $f(x)$ be defined for positive values of x. Then the integral $\int_0^\infty f(x)\cos sxdx$ is called the Fourier Cosine Transform of $f(x)$ and is denoted by $\hat{f}_c(s)$ or $F_c[f(x)]$. Thus

$$\hat{f}_c(s)=F_c\left[f(x)\right]=\frac{2}{\pi}\int_0^\infty f(x)\cos sxdx$$

The inverse Fourier Cosine Transform of $\hat{f}_c(s)$ is defined through the integral $\frac{2}{\pi}\int_0^\infty \hat{f}_c(s)\cos sxds$. This is denoted by $f(x)$ or $F_c^{-1}[\hat{f}_c(s)]$. Thus

$$f(x)=F_c^{-1}\left[\hat{f}(s)\right]=\frac{2}{\pi}\int_0^\infty \hat{f}_c(s)\cos s\,x\,d\,s$$

Basic Properties:

The following are the basic properties of cosine transforms:

(1) **Linearity property**: If 'a' and 'b' are two constants, then for two functions $f(x)$ and $\phi(x)$, we have $F_c\left[af(x)+b\phi(x)\right]=aF_c(f(x))+bF_c(\phi(x))$

(2) **Change of scale property**: If $F_c\{f(x)\}=\hat{f}_c(s)$, then for $a\neq 0$, we have $F_c\left[f(ax)\right]=\frac{1}{a}\hat{f}_c\left(\frac{s}{a}\right)$

(3) **Modulation property**: If $F_c\{f(x)\}=\hat{f}_c(s)$, then for $a\neq 0$, we have

$$F_c\left[f(x)\cos sx\right]=\frac{1}{2}\left[\hat{f}_c(s+a)+\hat{f}_c(s-a)\right]$$

The proofs of these properties are similar to the proofs of the corresponding properties of Fourier Sine Transforms.

EXAMPLES

1. Find the Fourier Sine transform of $e^{-|x|}$. Hence show that $\int_0^\infty \frac{x\sin mx}{1+x^2}dx=\frac{\pi e^{-m}}{2}, m>0$

<< $F_s[f(x)]=\int_0^\infty f(x)\sin sx dx=\int_0^\infty e^{-|x|}\sin sx dx=\int_0^\infty e^{-x}\sin sx dx$ [since |x| = - x in (0, ∞)]

$$=\frac{e^{-x}}{1+s^2}(-\sin sx-s\cos sx)\Big|_0^\infty=0-\frac{1}{1+s^2}(0-s)=\frac{s}{1+s^2}$$

By using inverse formula for Fourier sine transform, we get

$f(x)=\frac{2}{\pi}\int_0^\infty F_s[f(x)]\sin sx ds \Rightarrow e^{-x}=\frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}\sin sx ds$, By changing x to m, we get

$$e^{-m}=\frac{2}{\pi}\int_0^\infty \frac{s\sin ms}{1+s^2}ds=\frac{2}{\pi}\int_0^\infty \frac{x\sin mx}{1+x^2}dx \Rightarrow \int_0^\infty \frac{x\sin mx}{1+x^2}dx=\frac{\pi e^{-m}}{2}$$

2. Find the Fourier Cosine transform of $f(x)=\begin{cases} x & for \quad 0<x<1 \\ 2-x & for \quad 1<x<2 \\ 0 & for \quad x>2 \end{cases}$

<< $F_c[f(x)]=\int_0^\infty f(x)\cos sx dx=\int_0^1 f(x)\cos sx dx+\int_1^2 f(x)\cos sx dx+\int_2^\infty f(x)\cos sx dx$

$$=\int_0^1 x\cos sx dx+\int_1^2 (2-x)\cos sx dx+\int_2^\infty 0.\cos sx dx$$

$$=\left[x.\frac{\sin sx}{s}-1.\left(\frac{-\cos sx}{s^2}\right)\right]\Bigg|_0^1+\left[(2-x)\frac{\sin sx}{s}-(-1)\left(\frac{-\cos sx}{s^2}\right)\right]\Bigg|_1^2$$

$$=\frac{\sin s}{s}+\frac{\cos s}{s^2}-\left(0-\frac{1}{s^2}\right)-\frac{\cos 2s}{s^2}-\left(\frac{\sin s}{s}-\frac{\cos s}{s^2}\right)$$

$$=\frac{2\cos s}{s^2}-\frac{1}{s^2}-\frac{\cos 2s}{s^2}=\frac{1}{s^2}(2\cos s-\cos 2s-1)$$

3. Find the Fourier Sine transform of e^{-ax}/x

<< Let $f(x)=e^{-ax}/x$, then its Fourier sine transform is

$$F_s[f(x)]=\int_0^\infty f(x)\sin sx dx=\int_0^\infty \frac{e^{-ax}}{x}\sin sx dx=F(s), \text{say}$$

Differentiate both sides w.r.t. s we get,

$$\frac{d}{ds}\{F(s)\}=\int_0^\infty \frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x}\sin sx\right)dx=\int_0^\infty \frac{xe^{-ax}}{x}\cos sx dx=\int_0^\infty e^{-ax}\cos sx dx$$

$$=\frac{e^{-ax}}{a^2+s^2}(-a\cos sx+s\sin sx)\Big|_0^\infty$$

$$=\frac{1}{s^2+a^2}[0-(-a)]=\frac{a}{s^2+a^2}$$

Integrating w.r.t. s, we get,

$$F(s) = \int \frac{a}{s^2 + a^2} ds = \tan^{-1} \frac{s}{a} + c$$

But F(0) = 0 when s = 0 $\Rightarrow c = 0.$ Hence $F(s) = \tan^{-1} \frac{s}{a}$

4. Find the Fourier Cosine Transform of $f(x) = \frac{1}{1+x^2}$. Hence derive Fourier sine transform of $\phi(x) = \frac{x}{1+x^2}$

<< $F_c[f(x)] = \int_0^\infty f(x)\cos sx dx = \int_0^\infty \frac{1}{1+x^2}\cos sx dx = I(say)$

$$\therefore \frac{dI}{ds} = \int_0^\infty \frac{\partial}{\partial s}\left(\frac{\cos sx}{1+x^2}\right) dx = \int_0^\infty \frac{-x\sin sx}{1+x^2} dx = -\int_0^\infty \frac{x^2 \sin sx}{x(1+x^2)} dx = -\int \frac{[1+x^2-1]\sin sx}{x(1+x^2)} dx$$

$$= -\int_0^\infty \frac{\sin sx}{x} dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx \Rightarrow \frac{d^2 I}{ds^2} = \int_0^\infty \frac{x\cos sx}{x(1+x^2)} dx = I$$

$$\therefore \frac{d^2 I}{ds^2} - I = 0 \Rightarrow (D^2 - 1)I = 0, \text{where } D = \frac{d}{ds}$$

$$\therefore I = c_1 e^s + c_2 e^{-s} \Rightarrow \frac{dI}{ds} = c_1 e^s - c_2 e^{-s}$$

When $s = 0,$ we get $c_1 + c_2 = I = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$ & $c_1 - c_2 = \frac{dI}{ds} = -\frac{\pi}{2} \Rightarrow c_1 = 0$ & $c_2 = \frac{\pi}{2}$

$$\therefore F_c[f(x)] = I = \frac{\pi}{2} e^{-s}$$

Now $F_s[\phi(x)] = \int_0^\infty \frac{x\sin sx}{1+x^2} dx = -\frac{dI}{ds} = \frac{\pi}{2} e^{-s}$

5. Solve the integral equation $\int_0^\infty f(\theta)\cos\alpha\theta d\theta = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$.Hence evaluate $\int_0^\infty \frac{\sin^2 t}{t^2} dt$

<< We have $\int_0^\infty f(\theta)\cos\alpha\theta d\theta = F_c(\alpha)$

$$\therefore F_c(\alpha) = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases} \text{------------- (1)}$$

$\therefore$ by the inverse formula we have

$$f(\theta) = \frac{2}{\pi}\int_0^\infty F_c(\alpha)\cos\alpha\theta d\alpha = \frac{2}{\pi}\int_0^1 (1-\alpha)\cos\alpha\theta d\alpha = \frac{2}{\pi}\left[(1-\alpha)\frac{\sin\alpha\theta}{\theta} - (-1)\left(-\frac{\cos\alpha\theta}{\theta^2}\right)\right]_0^1$$

$$= \frac{2(1-\cos\theta)}{\pi\theta^2}$$

Now $F_c(\alpha) = \int_0^\infty f(\theta)\cos\alpha\theta\, d\theta = \int_0^\infty \frac{2(1-\cos\theta)}{\pi\theta^2}\cos\alpha\theta\, d\theta$ ----------- (2)

From (1) & (2) we have $\frac{2}{\pi}\int_0^\infty \frac{1-\cos\theta}{\theta^2}\cos\alpha\theta\, d\theta = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$

Now take limit as $\alpha \to 0$, we get

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos\theta}{\theta^2}\, d\theta = 1 \Rightarrow \int_0^\infty \frac{1-\cos\theta}{\theta^2}\, d\theta = \frac{\pi}{2} \Rightarrow \int_0^\infty \frac{2\sin^2\theta/2}{\theta^2}\, d\theta = \frac{\pi}{2}, \text{ put } \frac{\theta}{2} = t, \text{ then } d\theta = 2dt$$

$$\Rightarrow \int_0^\infty \frac{2\sin^2 t}{(t/2)^2} 2dt = \frac{\pi}{2} \Rightarrow \int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$$

(6) Solve the integral equation $\int_0^\infty f(x)\cos\alpha x dx = e^{-a\alpha}$

Let $\phi(\alpha)$ be defined by $\phi(\alpha) = e^{-a\alpha}$

Given $\phi(\alpha) = \int_0^\infty f(x)\cos\alpha x dx = \hat{f}_c(\alpha)$

Using this in the inversion formula, we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \phi(\alpha)\cos\alpha x d\alpha = \frac{2}{\pi}\int_0^\infty e^{-a\alpha}\cos\alpha x d\alpha$$

$$= \frac{2}{\pi}\left[\frac{e^{-a\alpha}}{a^2+x^2}\{-a\cos\alpha x + x\sin\alpha x\}\right]_0^\infty = \frac{2}{\pi}\left[\frac{a}{a^2+x^2}\right] = \frac{2a}{\pi(a^2+x^2)}$$

Exercises:

1. Find the Fourier sine transform of $f(x) = \begin{cases} 1, & 0 \le x \le a \\ 0, & x > a \end{cases}$ $\left[Ans: \frac{1-\cos sa}{s}\right]$

2. Find f(x) from the integral equation $\int_0^\infty f(x)\sin\alpha x dx = \begin{cases} 1, & 0 \le \alpha \le 1 \\ 2, & 1 \le \alpha < 2 \\ 0, & \alpha \ge 2 \end{cases}$

$\left[Ans: \frac{2}{\pi x}[1+\cos x - 2\cos 2x]\right]$

3. Find the sine transforms of the following functions

$(i)\ f(x) = \begin{cases} x, & 0 < x < 1 \\ a-x, & 1 < x < a \\ 0, & x > a \end{cases}$ $(ii)\ f(x) = xe^{-ax}, a > 0$ $(iii)\ f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$

(4) Solve for f(x) given $\int_0^\infty f(x)\sin\alpha x dx = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$

5) Find the inverse sine transforms of the following functions:

$(i)\ \hat{f}_s(s) = \frac{e^{-as}}{s},\ a > 0$ $(ii)\ \hat{f}_s(s) = \frac{\pi}{2}$

6. Find the Cosine transform of $f(x) = e^{-ax}, a > 0$. Hence evaluate $\int_0^\infty \frac{\cos kx}{x^2 + a^2} dx$

$$\left(Ans : \frac{a}{a^2 + \alpha^2} \ \&\ \int_0^\infty \frac{\cos kx}{x^2 + a^2} dx = \frac{\pi e^{-ax}}{2a} \right)$$

7. Find the Fourier Cosine Transforms of the following functions:

(i) $f(x) = \begin{cases} 4x, & 0 < x < 1 \\ 4 - x, & 1 < x < 4 \\ 0, & x > 4 \end{cases}$ $(ii)\ f(x) = e^{-ax^2},\ a > 0$ $(iii)\ f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$

$(iv)\ f(x) = xe^{-ax},\ a > 0$ $(v)\ f(x) = \frac{1}{1 + x^2}$ $(vi)\ f(x) = \frac{\cos 2x}{1 + x^2}$

8. Solve for $f(x)$ given $\int_0^\infty f(x) \cos \alpha x dx = \begin{cases} 1 - \alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$

Z – TRANSFORMS

Introduction

The Z-transform plays an important role in the study of communications, sample data control systems, discrete signal processing, solutions of difference equations etc.

Definition:

A difference equation is a relation between the differences of an unknown function at one or more general values of the argument.

Eg: $\Delta y_{(n+1)} + y_{(n)} = 2$ & $\Delta^2 y_{(n+1)} + \Delta y_{(n-1)} = 0$ are difference equations.

Another way of writing a difference equation is as follows:

We know that $\Delta y_{(n+1)} = y_{(n+2)} - y_{(n+1)}$

Hence the first equation can be written as $y_{(n+2)} - y_{(n+1)} + y_{(n)} = 2$ -------- (1)

Also $\Delta^2 y_{(n+1)} = y_{(n+3)} - 2y_{(n+2)} + y_{(n+1)}$

Hence the second equation can be written as

$y_{(n+3)} - 2y_{(n+2)} + y_{(n+1)} + y_{(n)} - y_{(n-1)} = 0$ --------- (2)

Order of a difference equation:

Order of a difference equation is the difference between the largest and the smallest arguments occurring in the difference equation divided by the unit of increment.

Thus the order of the equation (1) is 2. Since

$$\frac{Largest\ argument - smallest\ argument}{Unit\ of\ increment} = \frac{(n+2)-n}{1} = 2$$

& that of (2) is $\frac{(n+3)-(n-1)}{1} = 4$

Solution: Solution of a difference equation is an expression for $y_{(n)}$ which satisfies the given difference equation.

General solution: The general solution of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation.

A Particular solution or Particular integral is that solution which is obtained from the general solution by giving particular values to the constants.

Z – TRANSFORMS:

Definition: Let $u_n = f(n)$ be a real-valued function defined for $n = 0,1,2,3,\ldots\ldots\ldots$ and $u_n = 0$ for $n < 0$. Then the Z-transform of u_n denoted by $Z(u_n)$ is defined by

$U(z) = Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$ --------------- (1) whenever the infinite series converges

We can also write it as $Z^{-1}[U(z)] = u_n$ and is called the inverse Z – Transform.

Properties of Z-transform:

1. Linearity property:-

Consider the sequences $\{u_n\}$ & $\{v_n\}$ and constants a and b. Then

$Z[au_n + bv_n] = aZ[u_n] + bZ[v_n]$

Proof: By definition, we have $Z[au_n + bv_n] = \sum_{n=0}^{\infty}[au_n + bv_n]z^{-n}$

$$= a\sum_{n=0}^{\infty} u_n z^{-n} + b\sum_{n=0}^{\infty} v_n z^{-n} = aZ(u_n) + bZ(v_n)$$

In particular, for $a = b = 1$, we get $Z[u_n + v_n] = Z[u_n] + Z[v_n]$ and for $a = -b = 1$, we get

$Z[u_n - v_n] = Z[u_n] - Z[v_n]$

2. Damping property:-

Let $Z(u_n) = U(z)$. Then (i) $Z(a^n u_n) = U\left(z/a\right)$ (ii) $Z(a^{-n} u_n) = U(az)$

Proof: By definition, we have $Z(a^n u_n) = \sum_{n=0}^{\infty} (a^n u_n) z^{-n} = \sum_{n=0}^{\infty} u_n \left(\frac{z}{a}\right)^{-n} = U\left(\frac{z}{a}\right)$

Thus $Z(a^n u_n) = U\left(\frac{z}{a}\right)$

This is the result as desired. Here, we note that that if $Z(u_n) = U(z)$, then

$$Z(a^n u_n) = [U(z)]_{Z \to Z/a} = U\left(\frac{z}{a}\right)$$

Next, $Z(a^{-n} u_n) = \sum_{n=0}^{\infty} (a^{-n} u_n) z^{-n} = \sum_{n=0}^{\infty} u_n (az)^{-n} = U(az)$

Thus $Z(a^{-n} u_n) = U(az)$.This is the result as desired.

3. Shifting property:

(a) Right shifting rule:

If $Z(u_n) = U(z)$, then $Z(u_{n-k}) = z^{-k} U(z)$ where $k > 0$

Proof: By definition, we have $Z(u_{n-k}) = \sum_{n=0}^{\infty} u_{n-k} z^{-n}$

Since $u_n = 0$ for $n < 0$, we have $u_{n-k} = 0$ for $n = 0,1,2,\ldots\ldots\ldots,(k-1)$

Hence $Z(u_{n-k}) = \sum_{n=k}^{\infty} u_{n-k} z^{-n} = u_0 z^{-k} + u_1 z^{-(k+1)} + \ldots\ldots\infty = z^{-k} [u_0 + u_1 z^{-1} + \ldots\ldots + \infty]$

$$= z^{-k} \sum_{n=0}^{\infty} u_n z^{-n} = z^{-k} U(z)$$

Thus $Z(u_{n-k}) = z^{-k} U(z)$

(b)Left shifting rule: $Z(u_{n+k}) = z^k [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \ldots\ldots - u_{k-1} z^{-(k-1)}]$

Proof: $Z(u_{n+k}) = \sum_{n=0}^{\infty} u_{n+k} z^{-n} = z^k \sum_{n=0}^{\infty} u_{n+k} z^{-(k+n)} = z^k \left\{ \sum_{n=0}^{\infty} u_{n+k} z^{-(k+n)} \right\}$

$$= z^k \left[\sum_{m=k}^{\infty} u_m z^{-m} \right], \text{ where } \quad m = n + k$$

$$= z^k \left[\sum_{n=k}^{\infty} u_n z^{-n} \right] = z^k \left[\sum_{n=0}^{\infty} u_n z^{-n} - \sum_{n=0}^{k-1} u_n z^{-n} \right]$$

$$= z^k \left[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \ldots\ldots - u_{k-1} z^{-(k-1)} \right]$$

Particular cases:

In particular, we have the following standard results:

1. $Z(u_{n+1}) = z[U(z) - u_0]$

2. $Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$

3. $Z(u_{n+3}) = z^3[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$ etc.

Some Standard Z-Transforms:

1. Transform of a^n :

By definition, we have $Z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \ldots\ldots + \infty$

The series on the RHS is a Geometric series. Sum to infinity of the series is $\frac{1}{1 - a/z}$

or $\frac{z}{z-a}$ Thus, $Z(a^n) = \frac{z}{z-a}$

In particular, when a =1, we get Z(1) = $\frac{z}{z-1}$

2. Transform of e^{an} :

Here $Z(e^{an}) = Z(k^n)$ where $k = e^a$

$$= \frac{z}{z-k} = \frac{z}{z-e^a}$$

Thus $Z(e^{an}) = \frac{z}{z-e^a}$

3. Transform of n^p , p being a positive integer:

We have, $Z(n^p) = \sum_{n=0}^{\infty} n^p z^{-n} = z \sum_{n=0}^{\infty} n^{p-1} z^{-(n+1)} n$ -------- (1)

By changing p to $p-1$, we get $Z(n^{p-1}) = \sum_{n=0}^{\infty} n^{p-1} z^{-n}$

Differentiating with respect to z, we get

$$\frac{d}{dz}\left[Z(n^{p-1})\right] = \frac{d}{dz} \sum_{n=0}^{\infty} n^{p-1} z^{-n} = \sum_{n=0}^{\infty} n^{p-1}(-n) z^{-(n+1)}$$

Using this in (1), we get $Z(n^p) = -z\frac{d}{dz}[Z(n^{p-1})]$

Particular cases of $Z(n^p)$:-

1. For p = 1, we get $Z(n) = -z\frac{d}{dz}\left[Z(1)\right] = -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{z}{(z-1)^2}$

Thus, $Z(n) = \frac{z}{(z-1)^2}$

2. For p = 2, we get $Z(n^2) = -z\frac{d}{dz}\left[Z(n)\right] = -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right) = \frac{z^2+z}{(z-1)^3}$

Thus, $Z(n^2) = \dfrac{z^2 + z}{(z-1)^3}$

3. For p = 3, we get $Z(n^3) = \dfrac{z^3 + 4z^2 + z}{(z-1)^4}$ & soon

4. Transform of na^n

By damping property, we have

$$Z(na^n) = [Z(n)]_{Z \to Z/a} = \left[\frac{z}{(z-1)^2}\right]_{Z \to Z/a} = \frac{z/a}{\left(\frac{z}{a} - 1\right)^2} = \frac{az}{(z-a)^2}$$

Thus, $Z(na^n) = \dfrac{az}{(z-a)^2}$

5. Transform of $n^2 a^n$:

We have, $Z(n^2 a^n) = [Z(n^2)]_{Z \to \frac{Z}{a}} = \left[\dfrac{z^2 + z}{(z-1)^3}\right]_{Z \to \frac{Z}{a}} = \dfrac{(z/a)^2 + z/a}{(z/a - 1)^3} = \dfrac{(z^2 + az)a^3}{a^2(z-a)^3}$

Thus, $Z(n^2 a^n) = \dfrac{az^2 + a^2 z}{(z-a)^3}$

6. Transforms of $\cosh n\theta$ and $\sinh n\theta$:

We have $\cosh n\theta = \dfrac{e^{n\theta} + e^{-n\theta}}{2} \Rightarrow Z(\cosh n\theta) = \dfrac{1}{2} Z(e^{n\theta} + e^{-n\theta})$

$= \dfrac{1}{2}\left[Z(e^{n\theta}) + Z(e^{-n\theta})\right]$, by using the linearity property

$$= \frac{1}{2}\left[\frac{z}{z - e^{\theta}} + \frac{z}{z - e^{-\theta}}\right] = \frac{z}{2}\left[\frac{z - e^{-\theta} + z - e^{\theta}}{z^2 - z(e^{\theta} + e^{-\theta}) + 1}\right] = z\left[\frac{z - \frac{(e^{\theta} + e^{-\theta})}{2}}{z^2 - z(e^{\theta} + e^{-\theta}) + 1}\right]$$

$$= \frac{z[z - \cosh\theta]}{z^2 - 2z\cosh\theta + 1}$$

Next, $\sinh n\theta = \dfrac{e^{n\theta} - e^{-n\theta}}{2} \Rightarrow Z(\sinh n\theta) = \dfrac{z}{2}\left[\dfrac{1}{z - e^{\theta}} - \dfrac{1}{z - e^{-\theta}}\right]$

$$= \frac{z}{2}\left[\frac{e^{\theta} - e^{-\theta}}{z^2 - 2z\cosh\theta + 1}\right] = \frac{z\sinh\theta}{z^2 - 2z\cosh\theta + 1}$$

7. Transforms of $\cos n\theta$ and $\sin n\theta$

We have $\cos n\theta = \dfrac{e^{in\theta} + e^{-in\theta}}{2}$

$$Z(\cos n\theta) = \frac{1}{2} Z(e^{in\theta} + e^{-in\theta}) = \frac{1}{2}\left[\frac{z}{z-e^{i\theta}} + \frac{z}{z-e^{-i\theta}}\right] = \frac{z}{2}\left[\frac{2z-(e^{i\theta}+e^{-i\theta})}{z^2 - z(e^{i\theta}+e^{-i\theta})+1}\right]$$

$$= \frac{z[z-\cos\theta]}{z^2 - 2z\cos\theta + 1}$$

Next, $\sin n\theta = \dfrac{e^{in\theta} - e^{-in\theta}}{2i}$

$$Z(\sin n\theta) = \frac{1}{2i}\left[\frac{z}{z-e^{i\theta}} - \frac{z}{z-e^{-i\theta}}\right] = \frac{z}{2i}\left[\frac{e^{i\theta} - e^{-i\theta}}{z^2 - 2z\cos\theta + 1}\right] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

8. Transforms of $a^n \cos n\theta$ and $a^n \sin n\theta$

We know that $Z(\cos n\theta) = \dfrac{z[z-\cos\theta]}{z^2 - 2z\cos\theta + 1} = U(z)$. Hence by using damping rule we get

$$Z(a^n \cos n\theta) = U(z/a) = \frac{z/a[z/a - \cos\theta]}{(z/a)^2 - 2(z/a)\cos\theta + 1} = \frac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$$

Similarly we know that $Z(\sin n\theta) = \dfrac{z\sin\theta}{z^2 - 2z\cos\theta + 1} = U(z)$. Hence by using damping rule we get

$$Z(a^n \sin n\theta) = U(z/a) = \frac{(z/a)\sin\theta}{(z/a)^2 - 2(z/a)\cos\theta + 1} = \frac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$$

Examples: Find the Z-transforms of the following:

1. $3n - 4\sin(n\pi/4) + 5a$

<< $Z(3n - 4\sin(n\pi/4) + 5a) = 3Z(n) - 4Z(\sin(n\pi/4)) + 5aZ(1)$

$$= 3.\frac{z}{(z-1)^2} - 4.\frac{z\sin(\pi/4)}{z^2 - 2z\cos(\pi/4) + 1} + 5a.\frac{z}{z-1} = \frac{3z}{(z-1)^2} - \frac{4z\left(1/\sqrt{2}\right)}{z^2 - 2z\left(1/\sqrt{2}\right) + 1} + \frac{5az}{z-1}$$

$$= \frac{3z + 5az(z-1)}{(z-1)^2} - \frac{2\sqrt{2}z}{z^2 - \sqrt{2}z + 1} = \frac{(3-5a)z + 5az^2}{(z-1)^2} - \frac{2\sqrt{2}z}{z^2 - \sqrt{2}z + 1}$$

2. $(n+1)^2$

<< $Z[(n+1)^2] = Z(n^2 + 2n + 1) = Z(n^2) + 2Z(n) + Z(1) = \dfrac{z^2 + z}{(z-1)^3} + 2.\dfrac{z}{(z-1)^2} + \dfrac{z}{z-1}$

$$= \frac{z^2 + z + 2z(z-1) + z(z-1)^2}{(z-1)^3} = \frac{z^2 + z + 2z^2 - 2z + z^3 - 2z^2 + z}{(z-1)^3} = \frac{z^3 + z^2}{(z-1)^3}$$

3. $\sin(3n+5)$

<< $Z[\sin(3n+5)] = Z(\sin 3n \cos 5 + \cos 3n \sin 5) = \cos 5 Z(\sin 3n) + \sin 5 Z(\cos 3n)$

$$= \cos 5.\frac{z\sin 3}{z^2 - 2z\cos 3 + 1} + \sin 5.\frac{z(z-\cos 3)}{z^2 - 2z\cos 3 + 1} = \frac{z\sin 3\cos 5 + z^2\sin 5 - z\sin 5\cos 3}{z^2 - 2z\cos 3 + 1}$$

$$=\frac{z(\sin 3\cos 5-\cos 3\sin 5)+z^2\sin 5}{z^2-2z\cos 3+1}=\frac{z\sin(-2)+z^2\sin 5}{z^2-2z\cos 3+1}=\frac{z(z\sin 5-\sin 2)}{z^2-2z\cos 3+1}$$

4. ne^{an}

Let $u_n = n,\ e^{an} = k^n$ where $k = e^a$

Therefore $Z(ne^{an}) = Z(nk^n) = U(z/k)$, By damping rule.

Where $U(z) = Z(n) = \dfrac{z}{(z-1)^2} \Rightarrow U(z/k) = \dfrac{z/k}{(z/k-1)^2} = \dfrac{kz}{(z-k)^2}$, where $k = e^a$

Hence $Z(ne^{an}) = U(z/k)\dfrac{e^a z}{(z-e^a)^2}$

5. n^2e^{an}

Let $u_n = n^2,\ e^{an} = k^n$ where $k = e^a$

Therefore $Z(n^2e^{an}) = Z(n^2k^n) = U(z/k)$, By damping rule.

Where $U(z) = Z(n^2) = \dfrac{z^2+z}{(z-1)^3} \Rightarrow U(z/k) = \dfrac{(z/k)^2+z/k}{(z/k-1)^3} = \dfrac{kz^2+k^2z}{(z-k)^3}$, where $k = e^a$

Hence $Z(n^2e^{an}) = U(z/k)\dfrac{e^az^2+(e^a)^2z}{(z-e^a)^3} = \dfrac{e^az(z+e^a)}{(z-e^a)^3}$

6. i) $a^n\cosh n\theta$ ii) $a^n\sinh n\theta$

i) We know that $Z(\cosh n\theta) = \dfrac{z[z-\cosh\theta]}{z^2-2z\cosh\theta+1} = U(z)$, Hence by using damping rule we get $Z(a^n\cosh n\theta) = U(z/a) = \dfrac{(z/a)[(z/a)-\cosh\theta]}{(z/a)^2-2(z/a)\cosh\theta+1} = \dfrac{z(z-a\cosh\theta)}{z^2-2az\cosh\theta+a^2}$

ii) We know that $Z(\sinh n\theta) = \dfrac{z\sinh\theta}{z^2-2z\cosh\theta+1} = U(z)$, Hence by using damping rule we get $Z(a^n\sinh n\theta) = U(z/a) = \dfrac{(z/a)\cosh\theta}{(z/a)^2-2(z/a)\cosh\theta+1} = \dfrac{az\cosh\theta}{z^2-2az\cosh\theta+a^2}$

7. $e^t\sin 2t$

We know that $\sin 2t = \dfrac{z\sin 2}{z^2-2z\cos 2+1} = U(z)$

$\therefore Z(e^t\sin 2t) = U(z/e)$, By damping rule.

$$=\frac{(z/e)\sin 2}{(z/e)^2-2(z/e)\cos 2+1}=\frac{ez\sin 2}{z^2-2ez\cos 2+e^2}$$

8. $e^k\cos k\alpha\ (k \geq 0)$

We know that $Z(\cos k\alpha) = \dfrac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1} = U(z)$

$$\therefore Z(e^k \cos k\alpha) = Z((e^1)^k \cos k\alpha) = U(z/e) = \frac{(z/e)[(z/e)-\cos\alpha]}{(z/e)^2 - 2(z/e)\cos\alpha + 1} = \frac{z(z-e\cos\alpha)}{z^2 - 2ez\cos\alpha + e^2}$$

9. $\cos\left(\dfrac{n\pi}{2} + \dfrac{\pi}{4}\right)$

<< $Z\left(\cos\left(\dfrac{n\pi}{2}+\dfrac{\pi}{4}\right)\right) = Z\left(\cos\dfrac{n\pi}{2}\cos\dfrac{\pi}{4} - \sin\dfrac{n\pi}{2}\sin\dfrac{\pi}{4}\right) = \cos\dfrac{\pi}{4} Z\left(\cos\dfrac{n\pi}{2}\right) - \sin\dfrac{\pi}{4} Z\left(\sin\dfrac{n\pi}{2}\right)$

$$= \frac{1}{\sqrt{2}}\left(\frac{z(z-\cos\frac{\pi}{2}}{z^2 - 2z\cos\frac{\pi}{2} + 1}\right) - \frac{1}{\sqrt{2}}\left(\frac{z\sin\frac{\pi}{2}}{z^2 - 2z\cos\frac{\pi}{2} + 1}\right) = \frac{1}{\sqrt{2}}\left(\frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right) = \frac{z(z-1)}{\sqrt{2}(z^2+1)}$$

10. $\cosh\left(\dfrac{n\pi}{2} + \theta\right)$

<< $Z\left[\cosh\left(\dfrac{n\pi}{2}+\theta\right)\right] = Z\left[\dfrac{e^{n\pi/2+\theta} + e^{-(n\pi/2+\theta)}}{2}\right] = \dfrac{1}{2}\left[e^{\theta} Z\left(e^{n\pi/2}\right) + e^{-\theta} Z\left(e^{-(n\pi/2)}\right)\right]$

$= \dfrac{1}{2}\left[e^{\theta}\dfrac{z}{z-e^{\pi/2}} + e^{-\theta}\dfrac{z}{z-e^{-\pi/2}}\right]$ Using $Z(e^{ax}) = \dfrac{z}{z-e^a}$

$$= \frac{z}{2}\left[\frac{e^{\theta}(z-e^{-\pi/2}) + e^{-\theta}(z-e^{\pi/2})}{z^2 - z(e^{\pi/2}+e^{-\pi/2}) + 1}\right] = \frac{z}{2}\left[\frac{z(e^{\theta}+e^{-\theta}) - e^{-(\pi/2-\theta)} - e^{\pi/2-\theta}}{z^2 - z(e^{\pi/2}+e^{-\pi/2}) + 1}\right]$$

$$= \frac{z}{2}\left[\frac{z.2\cosh\theta - [e^{(\pi/2-\theta)} + e^{-(\pi/2-\theta)}}{z^2 - 2z\cosh(\pi/2) + 1}\right] = z\left[\frac{z\cosh\theta - \cosh(\pi/2-\theta)}{z^2 - 2z\cosh(\pi/2) + 1}\right]$$

$$= \frac{z^2\cosh\theta - z\cosh(\pi/2-\theta)}{z^2 - 2z\cosh(\pi/2) + 1}$$

11. ${}^nC_p \ (0 \le p \le n)$

<< $Z\left({}^nC_p\right) = \sum_{p=0}^{n}\left({}^nC_p z^{-p}\right) = 1 + {}^nC_1 z^{-1} + {}^nC_2 z^{-2} \ldots\ldots + {}^nC_n z^{-n} = (1+z^{-1})^n$

$$= \left(1+\frac{1}{z}\right)^n = \left(\frac{z+1}{z}\right)^n$$

12. ${}^{n+p}C_n$

$$Z\left({}^{n+p}C_n\right) = \sum_{p=0}^{\infty} {}^{n+p}C_n z^{-p} = 1 + {}^{n+1}C_n z^{-1} + {}^{n+2}C_n z^{-2} + {}^{n+3}C_n z^{-3} + \ldots\ldots\ldots$$

$= 1 + {}^{n+1}C_1 z^{-1} + {}^{n+2}C_2 z^{-2} + {}^{n+3}C_3 z^{-3} + \ldots\ldots\ldots,$ Using ${}^nC_{n-r} = {}^nC_r$

$$= 1 + (n+1)z^{-1} + \frac{(n+2)(n+1)}{2!}z^{-2} + \frac{(n+3)(n+2)(n+1)}{3!}z^{-3} + \ldots\ldots\ldots$$

$$=1+(-n-1)(-z)^{-1}+\frac{(-n-2)(-n-1)}{2!}(-z^{-1})^2+\frac{(-n-3)(-n-2)(-n-1)}{3!}(-z^{-1})^3+.......$$

$$=(1-z^{-1})^{-n-1}=\left(1-\frac{1}{z}\right)^{-(n+1)}=\left(\frac{z-1}{z}\right)^{-(n+1)}=\left(\frac{z}{z-1}\right)^{(n+1)}$$

13. Unit impulse sequence $\delta(n)=\begin{cases}1 & n=0\\ 0 & n\neq 0\end{cases}$

<< $Z[\delta(n)]=\sum_{n=0}^{\infty}\delta(n)z^{-n}=1+0+0+0+........=1$

14. Unit step sequence $u(n)=\begin{cases}0 & n<0\\ 1 & n\geq 0\end{cases}$

<< $Z[u(n)]=\sum_{n=0}^{\infty}u(n)z^{-n}=1+z^{-1}+z^{-2}+z^{-3}+......=\frac{1}{1-z^{-1}}=\frac{z}{z-1}$

15. Show that $Z\left(\frac{1}{n!}\right)=e^{1/z}$.Hence evaluate $Z\left(\frac{1}{(n+1)!}\right)$ & .$Z\left(\frac{1}{(n+2)!}\right)$

<< $Z\left(\frac{1}{n!}\right)=\sum_{n=0}^{\infty}\frac{1}{n!}.z^{-n}=1+\frac{z^{-1}}{1!}+\frac{z^{-2}}{2!}+\frac{z^{-3}}{3!}+......=e^{z^{-1}}=e^{1/z}$

Shifting $\frac{1}{n!}$ one unit to the left gives $Z\left(\frac{1}{(n+1)!}\right)=z[U(z)-u_0]=z\left[Z\left(\frac{1}{n!}\right)-1\right]=z\left(e^{1/z}-1\right)$

Similarly .$Z\left(\frac{1}{(n+2)!}\right)=z^2[U(z)-u_0-u_1z^{-1}]=z^2[e^{1/z}-1-z^{-1}]$

16. $u_n=\left(\frac{1}{2}\right)^n+\left(\frac{1}{4}\right)^n$

We have, $Z(u_n)=Z\left[\left(\frac{1}{2}\right)^n+\left(\frac{1}{4}\right)^n\right]=Z\left(\frac{1}{2}\right)^n+Z\left(\frac{1}{4}\right)^n=\frac{z}{z-\frac{1}{2}}+\frac{z}{z-\frac{1}{4}}=\frac{2z}{2z-1}+\frac{4z}{4z-1}$

$$=\frac{2z(8z-3)}{(2z-1)(4z-1)}$$

<u>Multiplication by n</u>: If $Z(u_n)=U(z)$, then $Z(nu_n)=-z\frac{d(U(z))}{dz}$

Proof: $Z(nu_n)=\sum_{n=0}^{\infty}nu_nz^{-n}=-z\sum_{n=0}^{\infty}u_n(-n)z^{-n-1}=-z\sum_{n=0}^{\infty}u_n\frac{d(z^{-n})}{dz}=-z\sum_{n=0}^{\infty}\frac{d(u_nz^{-n})}{dz}$

$$=-z\frac{d}{dz}\sum_{n=0}^{\infty}u_nz^{-n}=-z\frac{d(U(z))}{dz}$$

Note: In general $Z(n^mu_n)=(-z)^m\frac{d^m[U(z)]}{dz^m}$

Example: Find the z – transform of i) $n\sin n\theta$ ii) $n^2e^{n\theta}$

i) We know that $Z(\sin n\theta) = \dfrac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$, using $Z(nu_n) = -z\dfrac{d}{dz}U(z)$, we get

$$Z(n\sin n\theta) = -z\frac{d}{dz}\left(\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}\right) = -z\left(\frac{(z^2 - 2z\cos\theta + 1)\sin\theta - z\sin\theta(2z - 2\cos\theta)}{(z^2 - 2z\cos\theta + 1)^2}\right)$$

$$= -z\left(\frac{z^2\sin\theta - 2z\cos\theta\sin\theta + 1\sin\theta - 2z^2\sin\theta + 2z\sin\theta\cos\theta}{(z^2 - 2z\cos\theta + 1)^2}\right)$$

$$= -z\left(\frac{\sin\theta - z^2\sin\theta}{(z^2 - 2z\cos\theta + 1)^2}\right) = \frac{z(z^2 - 1)\sin\theta}{(z^2 - 2z\cos\theta + 1)^2}$$

ii) We know that $Z(e^{n\theta}) = \dfrac{z}{z - e^{\theta}}$ & $Z(n^2u_n) = (-z)^2\dfrac{d^2}{dz^2}U(z)$

$$\therefore Z(n^2e^{n\theta}) = (-z)^2\frac{d^2}{dz^2}\left(\frac{z}{z - e^{\theta})}\right) = z^2\frac{d}{dz}\left[\frac{(z - e^{\theta}).1 - z(1)}{(z - e^{\theta})^2}\right] = z^2\frac{d}{dz}\left[\frac{-e^{\theta}}{(z - e^{\theta})}\right]$$

$$= -z^2e^{\theta}\left(\frac{-2}{(z - e^{\theta})^3}\right) = \frac{2e^{\theta}z^2}{(z - e^{\theta})^3}$$

I. <u>Initial value theorem</u>: If $Z(u_n) = U(z)$, then $u_0 = \underset{z\to\infty}{Lt}\ U(z)$

Proof: we know that $U(z) = Z(u_n) = u_0 + u_1z^{-1} + u_2z^{-2} + ...$

Taking limit as $z \to \infty$, we get $\underset{z\to\infty}{Lt}\ U(z) = u_0$

Similarly $u_1 = \underset{z\to\infty}{Lt}\{z[U(z) - u_0]\}$, $u_2 = \underset{z\to\infty}{Lt}\{z^2[U(z) - u_0 - u_1z^{-1}]\}$, & so on

II. <u>Final value theorem</u>: If $Z(u_n) = U(z)$, then $\underset{n\to\infty}{Lt}(u_n) = \underset{z\to1}{Lt}(z-1)U(z)$

Proof: By definition $Z(u_{n+1} - u_n) = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$ Or

$$Z(u_{n+1}) - Z(u_n) = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

Or $Z[U(z) - u_0] - U(z) = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$ Or $U(z)(z-1) - u_0z = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$

Taking limits of both sides as $z \to 1$, we get

$$\underset{z\to1}{Lt}[(z-1)U(z)] - u_0 = \sum_{n=0}^{\infty}(u_{n+1} - u_n) = \underset{n\to\infty}{Lt}[(u_1 - u_0) + (u_2 - u_1) + (u_3 - u_2) + ... + (u_{n+1} - u_n)]$$

$$= \underset{n\to\infty}{Lt}\ u_{n+1} - u_0 = u_\infty - u_0$$

Hence $u_\infty = \underset{z\to1}{Lt}[(z-1)U(z)]$ Or $\underset{n\to\infty}{Lt}\ u_n = \underset{z\to1}{Lt}[(z-1)U(z)]$

Example: If $U(z)=\dfrac{2z^2+5z+14}{(z-1)^4}$, evaluate u_2 & u_3

<< $U(z)=\dfrac{2z^2+5z+14}{(z-1)^4}=\dfrac{1}{z^2}.\dfrac{2+5z^{-1}+14z^{-2}}{(1-z^{-1})^4}$,

By Initial value theorem, $u_0=\underset{z\to\infty}{Lt}\,U(z)=\underset{z\to\infty}{Lt}\left[\dfrac{1}{z^2}.\dfrac{2+5z^{-1}+14z^{-2}}{(1-z^{-1})^4}\right]=0$

Similarly $u_1=\underset{z\to\infty}{Lt}\{z[U(z)-u_0]\}=\underset{z\to\infty}{Lt}\,z\left[\dfrac{1}{z^2}.\dfrac{2+5z^{-1}+14z^{-2}}{(1-z^{-1})^4}\right]$

$$=\underset{z\to\infty}{Lt}\left[\frac{1}{z}.\frac{2+5z^{-1}+14z^{-2}}{(1-z^{-1})^4}\right]=0$$

Now $u_2=\underset{z\to\infty}{Lt}\{z^2[U(z)-u_0-u_1z^{-1}]\}=\underset{z\to\infty}{Lt}\left\{z^2\left[\dfrac{1}{z^2}.\dfrac{2+5z^{-1}+14z^{-2}}{(1-z^{-1})^4}\right]\right\}$

$$=\underset{z\to\infty}{Lt}\left[\frac{2+5z^{-1}+14z^{-2}}{(1-z^{-1})^4}\right]=2$$

& $u_3=\underset{z\to\infty}{Lt}\{z^3[U(z)-u_0-u_1z^{-1}-u_2z^{-2}]\}=\underset{z\to\infty}{Lt}\left\{z^3\left[\dfrac{2z^2+5z+14}{(z-1)^4}-0-0-2z^{-2}\right]\right\}$

$$=\underset{z\to\infty}{Lt}\left\{z^3\left[\frac{2z^2+5z+14}{(z-1)^4}-\frac{2}{z^2}\right]\right\}=\underset{z\to\infty}{Lt}\left\{z^3\left[\frac{z^2(2z^2+5z+14)-2(z-1)^4}{z^2(z-1)^4}\right]\right\}$$

$$=\underset{z\to\infty}{Lt}\left\{z^3\left[\frac{2z^4+5z^3+14z^2-2z^4-8z^2-2+8z^3+8z-4z^2}{z^2(z-1)^4}\right]\right\}$$

$$=\underset{z\to\infty}{Lt}\left\{z^3\left[\frac{13z^3+2z^2+8z-2}{z^2(z-1)^4}\right]\right\}=\underset{z\to\infty}{Lt}\left\{z^3\left[\frac{z^3[13+2z^{-1}+8z^{-2}-2z^{-3}]}{z^6(1-z^{-1})^4}\right]\right\}=13$$

Exercises: Find the Z – Transforms of the following:

1) $\dfrac{1}{(n+1)!}$ 2) $(\cos\theta+i\sin\theta)^n$ 3) $2n+5\sin\dfrac{n\pi}{4}-3a^4$ 4) $(n-1)^2$ 5) a^{n+3}

6) Show that i) $Z(e^{-an}\cos n\theta)=\dfrac{ze^a(ze^a-\cos\theta)}{z^2e^{2a}-2ze^a\cos\theta+1}$

ii) $Z(e^{-an}\sin n\theta)=\dfrac{ze^a\sin\theta}{z^2e^{2a}-2ze^a\cos\theta+1}$

7) Using $Z(n^2) = \dfrac{z^2+z}{(z-1)^3}$, Show that $Z(n+1)^2 = \dfrac{z^3+z^2}{(z-1)^3}$

8) If $Z(u_n) = \dfrac{z}{z-1} + \dfrac{z}{z^2+1}$, find the Z – Transform of u_{n+2}

9) *If* $U(z) = \dfrac{2z^2+3z+12}{(z-1)^4}$, find the value of u_2 & u_3

10) Given that $Z(u_n) = \dfrac{2z^2+3z+4}{(z-3)^3}$, $|z| > 3$, show that $u_1 = 2,\ u_2 = 21,\ u_3 = 139$

11) Show that $Z\left(\dfrac{1}{n}\right) = z \log \dfrac{z}{z-1}$

12) Using $Z(n) = \dfrac{z}{(z-1)^2}$, show that $Z(n\cos\theta) = \dfrac{(z^3+z)\cos\theta - 2z^2}{(z^2 - 2z\cos\theta + 1)^2}$

Answers: 1. $z(e^{1/z} - 1)$ 2. $\dfrac{z}{z-e^{i\theta}}$ 3. $\dfrac{2z}{(z-1)^2} + \dfrac{2/\sqrt{2}}{z^2-\sqrt{2}z+1} - \dfrac{3a^4z}{z-1}$

4. $\dfrac{z^3-3z^2+4z}{(z-1)^3}$ 5) $\dfrac{za^3}{z-a}$ 8) $\dfrac{z^2(1+3z^2)}{(1-z)(1+z^2)}$

9) $u_2 = 2,\ u_3 = 11$

<u>Inverse Z – Transform</u>: If $U(z)$ is Z – Transform of u_n, then u_n is called Inverse Z – Transform of $U(z)$, Denoted by $u_n = Z^{-1}[U(z)]$.

Some Useful Inverse Z – transforms:

1. $Z^{-1}\left[\dfrac{z}{z-a}\right] = a^n$

2. $Z^{-1}\left[\dfrac{z}{z-1}\right] = 1$

3. $Z^{-1}\left[\dfrac{az}{(z-a)^2}\right] = a^n.n$

4. $Z^{-1}\left[\dfrac{z}{(z-1)^2}\right] = n$

5. $Z^{-1}\left[\dfrac{az^2+a^2z}{(z-a)^3}\right] = a^n.n^2$

6. $Z^{-1}\left[\dfrac{z^2+z}{(z-1)^3}\right] = n^2$

7. $Z^{-1}\left[\dfrac{az^3+4a^2z^2+a^3z}{(z-a)^4}\right] = a^n.n^3$

8. $Z^{-1}\left[\dfrac{z^3+4z^2+z}{(z-1)^4}\right] = n^3$

9. $Z^{-1}\left[\dfrac{z}{z^2+1}\right] = \sin(n\pi/2)$

10. $Z^{-1}\left[\dfrac{z^2}{z^2+1}\right] = \cos(n\pi/2)$

<u>Evaluation of inverse Z – Transforms</u>: We have the following 3 methods –

1. Power series method:

Example: 1) Find the inverse Z – Transform of $\log \dfrac{z}{z+1}$, by power series method.

<< Putting $z = 1/y$, we get $U(z) = \log\frac{z}{z+1} = \log\frac{1/y}{1/y+1} = \log\frac{1}{1+y} = -\log(1+y)$

$$= -y + \frac{1}{2}y^2 - \frac{1}{3}y^3 + ...$$

$$= -z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{3}z^{-3} + ... = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^{-n}$$

$$\therefore u_n = \begin{cases} 0 & for\ \ n = 0 \\ (-1)^n/n & other\ wise \end{cases}$$

2) Find the inverse Z – Transform of $\frac{z}{(z+1)^2}$, by division method

$$z^{-1} - 2z^{-2} + 3z^{-3} - ....$$

<< By Dividing we get $z^2 + 2z + 1 \overline{)\, z}$

$$\begin{array}{l} z + 2 + z^{-1} \\ \hline -2 - z^{-1} \\ -2 - 4z^{-1} - 2z^{-3} \\ \hline 3z^{-1} + 2z^{-2} \\ 3z^{-1} + 6z^{-2} + 3z^{-3} \\ \hline -4z^{-2} - 3z^{-3} \end{array}$$

& so on

Hence $\frac{z}{(z+1)^2} = z^{-1} - 2z^{-2} + 3z^{-3} - 4z^{-4} + ...$, which is an infinite series

$$i.e.\ U(z) = \sum_{n=0}^{\infty}(-1)^{n-1}nz^{-n} \Rightarrow u_n = (-1)^n n$$

II. Partial fraction method:

1. Find the inverse Z – Transform of i) $\frac{2z^2 + 3z}{(z+2)(z-4)}$ ii) $\frac{z^3 - 20z}{(z-2)^3(z-4)}$

<< i) Write $U(z) = \frac{2z^2 + 3z}{(z+2)(z-4)}$ as $\frac{U(z)}{z} = \frac{2z+3}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4}$

Then $2z + 3 = A(z-4) + B(z+2) \Rightarrow 11 = 6B \Rightarrow B = 11/6$ & $-1 = -6A \Rightarrow A = 1/6$

$$\frac{U(z)}{z} = \frac{1}{6}\frac{1}{z+2} + \frac{11}{6}\frac{1}{z-4} \Rightarrow U(z) = \frac{1}{6}\frac{z}{z+2} + \frac{11}{6}\frac{z}{z-4}$$

On inversion we get $u_n = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n$

ii) Write $U(z)=\dfrac{z^3-20z}{(z-2)^3(z-4)}$ as $\dfrac{U(z)}{z}=\dfrac{z^2-20}{(z-2)^3(z-4)}=\dfrac{A+Bz+cz^2}{(z-2)^3}+\dfrac{D}{z-4}$

Then $(A+Bz+Cz^2)(z-4)+D(z-2)^3=z^2-20$

If $z=4,$ then $8D=16-20\Rightarrow D=-1/2$, By putting $z=0,1\,\&-1,$ we get

$-4A-8D=-20\Rightarrow A+2D=5\Rightarrow A-1=5\Rightarrow A=6$

$(\mathrm{A+B+C})(-3)-\mathrm{D}=1-20\Rightarrow -3\mathrm{A}-3\mathrm{B}-3\mathrm{C}-\mathrm{D}=-19\Rightarrow 3\mathrm{A}+3\mathrm{B}+3\mathrm{C}-1/2=19$

$6+\mathrm{B}+\mathrm{C}=\dfrac{19+1/2}{3}=\dfrac{39}{6}=\dfrac{13}{2}\Rightarrow B+C=\dfrac{13}{2}-6=\dfrac{1}{2}$ --------- (1)

& $(A-B+C)(-5)-27D=-19\Rightarrow -5(A-B+C)=-19+27(-1/2)=\dfrac{-38-27}{2}=\dfrac{-65}{2}$

$\Rightarrow A-B+C=\dfrac{-65}{2(-5)}=\dfrac{13}{2}\Rightarrow -B+C=\dfrac{13}{2}-6=\dfrac{1}{2}$ ------- (2)

By adding (1) & (2) we get

$2C=1\Rightarrow C=1/2\Rightarrow B=0$

$$\therefore \frac{U(z)}{z}=\frac{6+1/2z^2}{(z-2)^3}-\frac{1}{2}\frac{1}{z-4}\Rightarrow U(z)=\frac{6z+1/2z^3}{(z-2)^3}-\frac{1}{2}\frac{z}{z-4}=\frac{1}{2}\left[\frac{12z+z^3}{(z-2)^3}-\frac{z}{z-4}\right]$$

$$=\frac{1}{2}\left[\frac{z(z-2)^2+4z^2+8z}{(z-2)^3}-\frac{z}{z-4}\right]=\frac{1}{2}\left[\frac{z}{z-2}+2.\frac{2z^2+2^2z}{(z-2)^3}-\frac{z}{z-4}\right]$$

On inversion, we get $u_n=\dfrac{1}{2}\left[2^n+2.2^n.n^2-4^n\right]=2^{n-1}+n^2 2^n+2^{2n-1}$

2. Find the inverse Z – Transform of $\dfrac{2(z^2-5z+6.5)}{(z-2)(z-3)^2}$, for $2<|z|<3$

<<$U(z)=\dfrac{2(z^2-5z+6.5)}{(z-2)(z-3)^2}=\dfrac{A}{z-2}+\dfrac{B}{z-3}+\dfrac{C}{(z-3)^2}$, Then

$2(z^2-5z+6.5)=A(z-3)^2+B(z-2)(z-3)+C(z-2)$

$z=2\Rightarrow 2(4-10+6.5)=A\Rightarrow A=1,\ z=3\Rightarrow 2(9-15+6.5)=C\Rightarrow C=1$

$z=0\Rightarrow 13=9A+6B-2C,$ *i.e.* $13=9+6B-2\Rightarrow 6B=6\Rightarrow B=1$

$$\therefore U(z)=\frac{1}{z-2}+\frac{1}{z-3}+\frac{1}{(z-3)^2}=\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1}+\frac{1}{9}\left(1-\frac{z}{3}\right)^{-2}$$

(so that $2/z<1\,\&\,z/3<1$)

$$=\frac{1}{z}\left[1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+.......\right]-\frac{1}{3}\left[1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+.......\right]+\frac{1}{9}\left[1+\frac{2z}{3}+\frac{3z^2}{9}+\frac{4z^3}{27}+....\right]$$

Where $2<|z|<3$

$$= \left(\frac{1}{z} + \frac{2}{z^2} + \frac{2^2}{z^3} + \frac{2^3}{z^4} + ...\right) - \left(\frac{1}{3} + \frac{z}{3^2} + \frac{z^2}{3^3} + \frac{z^3}{3^4} + ...\right) + \left(\frac{1}{3^2} + \frac{2z}{3^3} + \frac{3z^2}{3^4} + \frac{4z^3}{3^5} + ...\right)$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n+1} z^n + \sum_{n=0}^{\infty} (n+1)\left(\frac{1}{3}\right)^{n+2} z^n$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \left[(n+1)\left(\frac{1}{3}\right)^{n+2} - \left(\frac{1}{3}\right)^{n+1}\right] z^n$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} (n-2)\left(\frac{1}{3}\right)^{n+2} z^n$$

On inversion, we get

$u_n = 2^{n-1}, n \geq 1 \,\&\, u_n = -(n+2)3^{n-2}, n \leq 0$

Exercises: Find the inverse Z – Transform of the following:

1. $\frac{z^2}{(z-1)(z-3)}$
2. $\frac{4z}{z-a}, |z| > |a|$
3. $\frac{5z}{(2-z)(3z-1)}$
4. $\frac{z}{(z-1)^2}$
5. $\frac{3z^2 + z}{(5z-1)(5z+2)}$
6. $\frac{8z - z^3}{(4-z)^3}$
7. $\frac{4z^2 - 2z}{z^3 - 5z^2 + 8z - 4}$
8. $\frac{z+3}{(z+1)(z-2)}$

Answers:

1. $\frac{1}{2}\left(3^{n+1} - 1\right)$ 2. $4a^n$ 3. $\left(\frac{1}{3}\right)^n - 2^n$ 4. n 5. $\frac{13}{75}(0.2)^n + \frac{4}{75}(-0.4)^n$

6. $(n^2 + 7n + 4)(4)^{n-1}$ 7. $2 + 2^n + 3(n-1)2^n, (n \geq 1)$ 8. $2(-i)^{n-1} - (-2)^{n-1}$

<u>Application to difference equations</u>:

Procedure to solve a linear difference equation with constant coefficients by Z 0 Transforms:

1. Take the Z – transforms of both sides of the difference equation.
2. Transpose all terms without $U(z)$ to the right.
3. Divide by the coefficient of $U(z)$, getting $U(z)$ as a function of z.
4. Find inverse Z – Transform to get u_n.

Example: 1) Solve $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$ with $u_0 = 0, u_1 = 1$

Solution: If $Z(u_n) = U(z)$, Then $Z(u_{n+1}) = z[U(z) - u_0] \,\&\, Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$

Also $Z(3^n) = \frac{z}{z-3}$

$\therefore$ Taking the Z – Transform on both sides, we get

$$Z[u_{n+2}]+4Z[u_{n+1}]+3Z[u_n]=Z\left[3^n\right]\Rightarrow z^2[U(z)-u_0-u_1z^{-1}]+4z[U(z)-u_0]+3U(z)=\frac{z}{z-3}$$

$\Rightarrow z^2[U(z)-0-1.z^{-1}]+4z[U(z)-0]+3U(z)=\dfrac{z}{z-3}$, (Using the given conditions).

$$i.e.\ (z^2+4z+3)U(z)=z+\frac{z}{z-3}=\frac{z^2-3z+z}{z-3}=\frac{z^2-2z}{z-3}$$

$$\therefore U(z)=\frac{z^2-2z}{(z^2+4z+3)(z-3)}=\frac{z^2-2z}{(z+1)(z+3)(z-3)}$$

Let $\dfrac{U(z)}{z}=\dfrac{z-2}{(z+1)(z+3)(z-3)}=\dfrac{A}{z+1}+\dfrac{B}{z+3}+\dfrac{C}{z-3}$

$\Rightarrow z-2=A(z+3)(z-3)+B(z+1)(z-3)+C(z+1)(z+3)$

$z=-3\Rightarrow -5=B(-2)(-6)\Rightarrow B=-\dfrac{5}{12},\ z=3\Rightarrow 1=C(4)(6)\Rightarrow C=\dfrac{1}{24},$

$z=-1\Rightarrow -3=A(2)(-4)\Rightarrow A=\dfrac{3}{8}.$

$\therefore \dfrac{U(z)}{z}=\dfrac{3}{8}\dfrac{1}{z+1}-\dfrac{5}{12}\dfrac{1}{z+3}+\dfrac{1}{24}\dfrac{1}{z-3}\Rightarrow U(z)=\dfrac{3}{8}\dfrac{z}{z+1}-\dfrac{5}{12}\dfrac{z}{z+3}+\dfrac{1}{24}\dfrac{z}{z-3},$

On inversion we get,

$$u_n=\frac{3}{8}Z^{-1}\left[\frac{z}{z+1}\right]-\frac{5}{12}Z^{-1}\left[\frac{z}{z+3}\right]+\frac{1}{24}Z^{-1}\left[\frac{z}{z-3}\right]=\frac{3}{8}(-1)^n-\frac{5}{12}(-3)^n+\frac{1}{24}3^n$$

2) Solve $y_{n+2}+6y_{n+1}+9y_n=2^n$ with $y_0=y_1=0$, using Z – Transform.

Solution: Let $Z(y_n)=U(z)$, then $Z(y_{n+1})=z[U(z)-y_0]$ & $Z(y_{n+2})=z^2[U(z)-y_0-y_1z^{-1}]$

Also $Z(2^n)=\dfrac{z}{z-2}$. $\therefore$ by taking Z – Transform we get,

$$Z[y_{n+2}]+6Z[y_{n+1}]+9Z[y_n]=Z\left[2^n\right]\Rightarrow z^2[U(z)-y_0-y_1z^{-1}]+6z[U(z)-y_0]+9U(z)=\frac{z}{z-2}$$

$i.e.\ z^2[U(z)-0-0.z^{-1}]+6z[U(z)-0]+9U(z)=\dfrac{z}{z-2}$ (Using the condition $y_0=y_1=0$)

$$\Rightarrow (z^2+6z+9)U(z)=\frac{z}{z-2}\Rightarrow U(z)=\frac{z}{(z^2+6z+9)(z-2)}$$

Let $\dfrac{U(z)}{z}=\dfrac{1}{(z^2+6z+9)(z-2)}=\dfrac{1}{(z+3)^2(z-2)}=\dfrac{A}{z+3}+\dfrac{B}{(z+3)^2}+\dfrac{C}{z-2}$

$\therefore 1=A(z+3)(z-2)+B(z-2)+C(z+3)^2$

$z=2\Rightarrow 1=C(25)\Rightarrow C=\dfrac{1}{25},\ z=-3\Rightarrow 1=B(-5)\Rightarrow B=-\dfrac{1}{5}$ & $z=0\Rightarrow 1=-6A-2B+9C$

$$\Rightarrow 1 = -6A - 2\left(-\frac{1}{5}\right) + 9\left(\frac{1}{25}\right) \Rightarrow 6A = \frac{9}{25} + \frac{2}{5} - 1 = \frac{9+10-25}{25} = -\frac{6}{25} \Rightarrow A = -\frac{1}{25}$$

$$\therefore \frac{U(z)}{z} = -\frac{1}{25}\frac{1}{z+3} - \frac{1}{5}\frac{1}{(z+3)^2} + \frac{1}{25}\frac{1}{z-2} = \frac{1}{25}\left[\frac{1}{z-2} - \frac{1}{z+3} - \frac{5}{(z+3)^2}\right]$$

$\Rightarrow U(z) = \frac{1}{25}\left[\frac{z}{z-2} - \frac{z}{z+3} - \frac{5z}{(z+3)^2}\right]$, On taking inverse Z – transform we get,

$$y_n = \frac{1}{25}\left[Z^{-1}\left[\frac{z}{z-2}\right] - Z^{-1}\left[\frac{z}{z+3}\right] + \frac{5}{3}Z^{-1}\left[\frac{-3z}{(z+3)^2}\right]\right]$$

$$= \frac{1}{25}\left[2^n - (-3)^n + \frac{5}{3}\left(n(-3)^n\right)\right] \quad \left[\because Z^{-1}\left[\frac{az}{(z-a)^2}\right] = na^n\right]$$

3) Find the response of the system $y_{n+2} - 5y_{n+1} + 6y_n = u_n$ with $y_0 = 0,\ y_1 = 1$ & $u_n = 1$ for $n = 0,1,2,........$ by Z – Transform method.

Solution: Taking Z – Transform of both sides we get,

$$Z[y_{n+2}] - 5Z[y_{n+1}] + 6Z[y_n] = Z[u_n] = Z[1]$$

i.e. $z^2[U(z) - y_0 - y_1 z^{-1}] - 5z[U(z) - y_0] + 6U(z) = \frac{z}{z-1}$, Using $y_0 = 0$ & $y_1 = 1$ we get,

$$z^2[U(z) - 0 - 1.z^{-1}] - 5z[U(z) - 0] + 6U(z) = \frac{z}{z-1}$$

$$\Rightarrow (z^2 - 5z + 6)U(z) - z = \frac{z}{z-1} \Rightarrow (z^2 - 5z + 6)U(z) = z + \frac{z}{z-1} = \frac{z^2 - z + z}{z-1} = \frac{z^2}{z-1}$$

$$\Rightarrow U(z) = \frac{z^2}{(z-1)(z^2 - 5z + 6)} = \frac{z^2}{(z-1)(z-2)(z-3)}$$

Let $\frac{U(z)}{z} = \frac{z}{(z-1)(z-2)(z-3)} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3}$

$$\Rightarrow z = A(z-2)(z-3) + B(z-1)(z-3) + C(z-1)(z-2)$$

$$z = 1 \Rightarrow 1 = A(-1)(-2) \Rightarrow A = \frac{1}{2}, \quad z = 2 \Rightarrow 2 = B(1)(-1) \Rightarrow B = -2,$$

$$z = 3 \Rightarrow 3 = C(2)(1) \Rightarrow C = \frac{3}{2}$$

$\therefore \frac{U(z)}{z} = \frac{1}{2}\frac{1}{z-1} - \frac{2}{z-2} + \frac{3}{2}\frac{1}{z-3} \Rightarrow U(z) = \frac{1}{2}\frac{z}{z-1} - 2\frac{z}{z-2} + \frac{3}{2}\frac{z}{z-3}$, inversion we get

$$y_n = \frac{1}{2}Z^{-1}\left[\frac{z}{z-1}\right] - 2Z^{-1}\left[\frac{z}{z-2}\right] + \frac{3}{2}Z^{-1}\left[\frac{z}{z-3}\right] = \frac{1}{2}(1)^n - 2.2^n + \frac{3}{2}3^n = 2^{-1} - 2^{n+1} + 3^{n+1}2^{-1}$$

4. Using the Z – Transform, solve $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$

Solution: Taking Z – Transform of both sides we get,

$$Z[u_{n+2}] - 2Z[u_{n+1}] + Z[u_n] = Z[3n+5]$$

$$\Rightarrow z^2[U(z)-u_0-u_1z^{-1}]-2z[U(z)-u_0]+U(z)=3\frac{z}{(z-1)^2}+5\frac{z}{z-1}$$

$$\Rightarrow (z^2-2z+1)U(z)-z^2u_0-u_1z+2zu_0=\frac{3z}{(z-1)^2}+\frac{5z}{z-1}$$

$$\Rightarrow (z^2-2z+1)U(z)=\frac{3z+5z^2-5z}{(z-1)^2}+(z^2-2z)u_0+u_1z=\frac{5z^2-2z}{(z-1)^2}+(z^2-2z)u_0+u_1z$$

$\Rightarrow U(z)=\dfrac{5z^2-2z}{(z-1)^4}+\dfrac{(z^2-2z)}{(z-1)^2}u_0+u_1\dfrac{z}{(z-1)^2}$, On inversion we get,

$u_n=Z^{-1}\left[\dfrac{5z^2-2z}{(z-1)^4}\right]+u_0Z^{-1}\left[\dfrac{(z^2-2z)}{(z-1)^2}\right]+u_1Z^{-1}\left[\dfrac{z}{(z-1)^2}\right]$ ------- (I)

Now we know that $Z^{-1}[1]=\dfrac{z}{z-1}$, $Z^{-1}[n]=\dfrac{z}{(z-1)^2}$, $Z^{-1}[n^2]=\dfrac{z^2+z}{(z-1)^3}$ &

$Z^{-1}[n^3]=\dfrac{z^3+4z^2-z}{(z-1)^4}$.

$\therefore$ We can write $\dfrac{5z^2-2z}{(z-1)^4}$ as

$$\frac{5z^2-2z}{(z-1)^4}=A\frac{z^3+4z-z}{(z-1)^4}+B\frac{z^2+z}{(z-1)^3}+C\frac{z}{(z-1)^2}+D\frac{z}{z-1}$$

$$\Rightarrow 5z^2-2z=A(z^3+4z^2-z)+B(z^2+z)(z-1)+Cz(z-1)^2+Dz(z-1)^3$$

i.e. $5z^2-2z=A(z^3+4z^2-z)+B(z^3-z)+C(z^3-2z^2+z)+D(z^4-3z^3+3z^2-z)$

By equating coefficient of like powers of z, we get

$D=0$, $A+B+C-3D=0$, $4A-2C+3D=5$, $-A-B+C-D=-2$

$D=0\Rightarrow A+B+C=0$ ------- (1)

$4A-2C=5$ -------- (2)

$-A-B+C=-2$ ------ (3)

Adding (1) & (3) we get $2C=-2\Rightarrow C=-1$

From (2) we get $4A-2(-1)=5\Rightarrow 4A=5-2=3\Rightarrow A=\dfrac{3}{4}$.

From (3) we get $B=-A+C+2=-\dfrac{3}{4}-1+2=\dfrac{-3-4+8}{4}=\dfrac{1}{4}$.

$$\therefore \frac{5z^2-2z}{(z-1)^4}=\frac{3}{4}\frac{z^3+4z-z}{(z-1)^4}+\frac{1}{4}\frac{z^2+z}{(z-1)^3}-\frac{z}{(z-1)^2}$$

Hence $Z^{-1}\left[\dfrac{5z^2-2z}{(z-1)^4}\right]=\dfrac{3}{4}Z^{-1}\left[\dfrac{z^3+4z-z}{(z-1)^4}\right]+\dfrac{1}{4}Z^{-1}\left[\dfrac{z^2+z}{(z-1)^3}\right]-Z^{-1}\left[\dfrac{z}{(z-1)^2}\right]$

$$= \frac{3}{4}n^3 + \frac{1}{4}n^2 - n = \frac{1}{4}\left(3n^3 + n^2 - 4n\right) = \frac{n}{4}\left(3n^2 + n - 4\right) = \frac{1}{4}n(n-1)(3n+4)$$

$$\frac{z-2}{(z-1)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2} \Rightarrow z - 2 = A(z-1) + B$$

$z = 1 \Rightarrow -1 = B \Rightarrow B = -1$ & $z = 0 \Rightarrow -2 = -A + B \Rightarrow A = B + 2 = -1 + 2 = 1$

$$\therefore \frac{z-2}{(z-1)^2} = \frac{1}{z-1} - \frac{1}{(z-1)^2} \Rightarrow \frac{z^2 - 2z}{(z-1)^2} = \frac{z}{z-1} - \frac{z}{(z-1)^2}$$

$$\therefore Z^{-1}\left[\frac{z^2 - 2z}{(z-1)^2}\right] = Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z}{(z-1)^2}\right] = 1 - n \ \ \& \ Z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$$

Substituting these values in (I) we get

$$u_n = \frac{1}{4}n(n-1)(3n+4) + u_0(1-n) + u_1 n = \frac{1}{4}n(n-1)(3n+4) + u_0 + (u_1 - u_0)n$$

$\frac{1}{4}n(n-1)(3n+4) + c_0 + c_1 n,$ Where $c_0 = u_0$ & $c_1 = u_1 - u_0$

Exercises: Solve the following difference equations using Z – Transforms

1. $y_{n+2} - 4y_n = 0$ given that $y_0 = 0, y_1 = 2$
2. $u_{n+2} - 5u_{n+1} + 6u_n = 0$
3. $x(n+2) - 3x(n+1) + 2x(n) = 0,$ given that $x(0) = 0, x(1) = 1$
4. $f(n) = 3f(n-1) - 4f(n-2) = 0, n \geq 2,$ given that $f(0) = 3, f(1) = -2$
5. $u_{n+2} - 5u_{n+1} - 6u_n = 2^n$
6. $y_{n+2} - 6y_{n+1} + 9y_n = 3^n$
7. $y_{n+1} + \frac{1}{4}y_n = \left(\frac{1}{4}\right)^n \ (n \geq 0), y_0 = 0$
8. $y_{n+2} + 2y_{n+1} + y_n = n$ with $y_0 = y_1 = 0$
9. $u_{k+2} - 2u_{k+1} + u_k = 2^k$ with $u_0 = 2, u_1 = 1$
10. $y_{n+2} - 5y_{n+1} - 6y_n = 4^n$

Answers: 1. $y_n = 2^{n-1} + (-2)^{n-1}$ 2. $u_n = c_1.2^n + c_2.3^n$ 3. $x(n) = 1 - 2^n$

4. $f(n) = 2 + (-4)^n$ 5. $u_n = c_1(-1)^n + c_2(6)^n - \frac{1}{12}(2)^n$

6. $y_n = (c_1 + c_2 n)3^n + \frac{1}{2}n(n-1)3^{n-2}$ 7. $y_n = 2\left[\left(\frac{1}{4}\right)^n - \left(-\frac{1}{4}\right)^n\right]$

8. $y_n = \frac{3}{4}n$ 9. $u_k = 1 - 2k + 2^k$ 10. $y_n = c_1(6)^n + c_2(-1)^n - \frac{1}{10}4^n$

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Module-5 – Probability Distributions

- Review of basic probability theory, Random variables-discrete and continuous Probability distribution function, cumulative distribution function, Mathematical Expectation, mean and variance, Binomial, Poisson,Exponential and Normal distribution (without proofs for mean and SD) – Problems.
- **Sampling Theory:** Introduction to sampling distributions, standard error, Type-I and Type-II errors.Student's t-distribution, Chi-square distribution as a test of goodness of fit.
- **Self-study:** Test of hypothesis for means, single proportions only.

Probability Distributions

Random Variable: A random variable is a rule which assigns a numerical value to each and every outcomes of the random experiment. It is nothing but a function from the sample space 'S' to the set of all real numbers, denoted as $f : S \to R$. Random Variables are usually denoted by X, Y, Z, ………. The set of all real numbers of a random variable X is called the range of X.

Example:

1. While tossing a coin, suppose the value 1 is assigned for the outcome Head (H) & 0 is assigned for Tail (T), then S = {H, T), X(H) =1 and X(T) = 0, then the range of X = {0, 1}

2) Tossing 3 fair coins up, Then S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

Now X(HHH}= 3, X(HHT) = X(HTH} = X(THH} = 2, X(TTH} = X(THT} = X(HTT} = 1 &

X(TTT} = 0 & Range of X = {0, 1, 2, 3}

3) Let the random experiment be throwing a pair of 'die' and the sample space S associated with it is the set of all pair of numbers chosen from 1 to 6. i.e. S = {(x, y) / x, y $\in$ {1, 2, 3, 4, 5, 6}}

Then to each outcome (x, y) of S let us associate a random variable X = x + y

Now S = {(1, 1), (1, 2), ………., (2, 1), (2, 2), ………, (6, 5), (6, 6)}

Therefore X(1, 1) = 2, X(1, 2) = X(2, 1) = 3, ……….., X(6, 6) = 12 $\Rightarrow$ Range = {2, 3, 4, …….,12}

There are two types of Random Variable:

1. Discrete Random Variable   2. Continuous Random Variables

A variable X which takes finite or countably infinite number of values is called discrete random variable

Example: 1. throwing a die , random variable X is the number obtained

i.e. X(S) = {1, 2 , 3 , 4, 5, 6}, X is discrete random variable.(or whole number)

2. X is the total marks scored by a student in an examination (whole number)

A random variable whose range is uncountably infinite is called random variable. Here the range of variable is an interval of real numbers.

Example: 1. weight of articles

2. Observing the pointer on a speedometer / voltmeter

3. Conducting a survey on the life of electric bulbs.

According to the type of random variable we have two types of probability distributions

1. Discrete probability distribution   2. Continuous probability distribution.

1. Discrete Probability Distribution:

Probability mass function (p.m.f):

Let X be a discrete random variable and $p(x_i) = P[X = x_i]$, then $p(x_i)$ is the probability mass function (p.m.f.) of X if $(i)\ p(x_i) \geq 0$ for all x_i $(ii) \sum_i p(x_i) = 1$. *i.e.* $p(x_1) + p(x_2) + p(x_3) + ... + p(x_n) = 1$

Discrete Probability Distribution: It is a systematic presentation of value taken by the random variable with the corresponding probabilities**.**

The set of values $[x_i, p(x_i)]$ is called a discrete probability distribution of the discrete random variable X.

The distribution function F(x) of the discrete random variable X is defined as

$F(x) = P(X \leq x) = \sum_{i=1}^{x} p(x_i),$ where x is any integer. Also it is known as **cumulative distribution function (c. d. f.)**

Example: The discrete probability distribution for X, the sum of the numbers which turn on tossing a pair of dice is

$X = x_i$	2	3	4	5	6	7	8	9	10	11	12
$p(x_i)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Note That $p(x_i) \geq 0\ \&\ \sum_i p(x_i) = 1$

2. Continuous Probability Distribution:

Probability density function (p.d.f.): Let X be a continuous random variable. Then a function f(x) is a p.d.f. of X if

(i) $f(x) \geq 0,$ $(ii) \int_{-\infty}^{\infty} f(x)dx = 1$

The distribution function F(x) of the continuous random variable X is defined as

$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$. Also it is known as **cumulative distribution function (c. d. f.)**

Properties of cumulative distribution function:

i) $F^1(x) = f(x) \geq 0, \Rightarrow$ F(x) is a non – decreasing function. $ii)\ F(-\infty) = 0;\quad iii)\ F(\infty) = 1$

iv) If r is any real number, then $P(X \geq r) = \int_r^\infty f(x)dx$ &

$$P(X < r) = 1 - P(X \geq r)\ i.e.\ P(X < r) = 1 - \int_r^\infty f(x)dx$$

$$v)\ P(a \leq X \leq b) = \int_a^b f(x)dx = \int_a^{-\infty} f(x)dx + \int_{-\infty}^b f(x)dx = \int_{-\infty}^b f(x)dx - \int_{-\infty}^a f(x)dx = F(b) - F(a)$$

Expectation: The mean value (μ) of the probability distribution of a random variable (variate) X is commonly known as its expectation and is denoted by E(X). If $p(x_i)$ is the probability mass function of the variable X, then $E(X) = \sum_i x_i p(x_i)$ (for discrete distribution)

If $f(x)$ is the probability density function of the variable X, then $E(X) = \int_{-\infty}^{\infty} xf(x)dx$ (for continuous distribution)

Variance: Variance of a distribution is given by

$\sigma^2 = \sum_i (x_i - \mu)^2 p(x_i) = E(X^2) - (E(X))^2$ (for discrete distribution) where $E(X^2) = \sum_i x_i^2 p(x_i)$

& $\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx = E(X^2) - (E(X))^2$ (for continuous distribution) where

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$$

Note that σ is standard deviation of the distribution.

Problems:

1. A coin is tossed twice. A random variable X represents the number of heads turning up. Find the discrete probability distribution for X. Also find its mean & variance. [Ans: mean = 1, variance = ½]
2. A random experiment of tossing a ‘die’ twice is performed. Random variable X on this sample space is defined to be the sum of the two numbers turning up on the toss. Find the discrete probability distribution for the random variable X and compute the corresponding mean & standard deviation. [Ans: mean = 7, $S.D. = \sqrt{35/6} = 2.4152$]

3. Show that the following distribution represents a discrete probability distribution. Find the mean & variance.

x_i :	10	20	30	40
$p(x_i)$:	1/8	3/8	3/8	1/8

[Ans: mean = 25& variance = 75]

4. For the following function

$X = x_i$	-2	-1	0	1	2	3
$p(x_i)$	0.1	k	0.2	2k	0.3	k

Find (i) k, (ii) E(X) & (iii) Var(X) [Ans: (i) k = 0.1 (ii) E(X) = 0.8 (iii) E(X) = 2.8, V(X) = 2.16]

5. For the following function

x_i	0	1	2	3	4
$p(x_i)$	0.2	0.35	0.25	0.15	0.05

Find E(X) & V(X) [Ans: E(X) = 1.5, V(X) = 1.25]

6. A random variable X has the following probability function for various values of x.

$X = x_i$	0	1	2	3	4	5	6	7
	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2 + k$

Find (i) k (ii) E(X) and (iii) P(X < 6) [Ans: (i) k = 1/10 (ii) E(X) = 3.66 (iii) P(X < 6) = 0.81]

7. Find which of the following function is a probability density function.

$$(i)\ f_1(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & other\ wise \end{cases} \qquad (ii)\ f_2(x) = \begin{cases} 2x & -1 < x < 1 \\ 0 & other\ wise \end{cases}$$

$$(iii)\ f_3(x) = \begin{cases} |x| & |x| \le 1 \\ 0 & other\ wise \end{cases} \qquad (iv)\ f_4(x) = \begin{cases} 2x & 0 < x \le 1 \\ 4 - 4x & 1 < x < 2 \\ 0 & other\ wise \end{cases}$$

Solution: Condition for a p. d. f. are $f(x) \ge 0\ \&\ \int_{-\infty}^{\infty} f(x)dx = 1$

(i) $Here\ f(x)\geq 0\ \&\ \int_{-\infty}^{\infty} f_1(x)dx = \int_0^1 2xdx = 2\left[\frac{x^2}{2}\right]_0^1 = 1$. Hence $f_1(x)$ is p. d. f.

(ii) $f_2(x)$ can be written in the form $f_2(x)=\begin{cases} 2x & -1<x<0 \\ 2x & 0<x<1 \\ 0 & otherwise \end{cases}$

In $-1<x<0,\ f_2(x)=2x<0$ & $\int_{-\infty}^{\infty} f_2(x)dx = \int_{-1}^{1} 2xdx = \left[x^2\right]_{-1}^{1} = 0$

Hence both the conditions are not satisfied $\Rightarrow f_2(x)$ is not a p. d. f.

(iii) $f_3(x)=|x|\geq 0$ & $\int_{-\infty}^{\infty} f_3(x)dx = \int_{-1}^{1}|x|\,dx = \int_{-1}^{0}|x|\,dx + \int_{0}^{1}|x|\,dx$

$$= \int_{-1}^{0} -xdx + \int_0^1 xdx = -\left[\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}+\frac{1}{2} = 1.\ \therefore\ f_3(x) \text{ is a p. d. f.}$$

iv) $f_4(x)=2x>0\ \ in\ \ 0<x\leq 1$. But $f_4(x)=4-4x<0\ \ in\ \ 1<x<2$

The first condition is not satisfied $\Rightarrow f_4(x)$ is not a p. d. f.

8. Find the constant k such that $f(x)=\begin{cases} kx^2, & 0<x<3 \\ 0, & otherwise \end{cases}$ is a p. d. f.

Also compute $(i)\ P(1<X<2),\ (ii)\ P(X\leq 1)\ \ (iii)\ P(X>1)$ (iv) Mean (v) Variance.

[Ans: k = 1/9 $(i)\ P(1<X<2)=7/27$ $(ii)\ P(X\leq 1)=1/27$ $(iii)\ P(X>1)=26/27$ (iv) Mean = 2.25

(v) $E(X^2)$ = 5.4, Variance = 0.3375]

9. The p. d. f. of a continuous random variable is given by $f(x)=\begin{cases} kxe^{-x}; & 0<x<1 \\ 0 & other\ wise \end{cases}$

Find k and hence find the mean.

[Answer: $k=\frac{e}{e-2}=3.7844,\ mean=\frac{2e-5}{e-2}=0.6078$

10. A random variable X has the following density function, $f(x)=\begin{cases} kx^2; & -3\leq x\leq 3 \\ 0 & otherwise \end{cases}$ Evaluate k &

find (i) $P(1\leq x\leq 2)$ (ii) $P(x\leq 2)$ (iii) $P(x>1)$ [Ans: k = 1/18, (i) 7/54 (ii) 35/54 (iii) 13/27]

11. A continuous random variable has the distribution function $F(x)=\begin{cases} 0 & x\le 1 \\ c(x-1)^4, & 1\le x\le 3 \\ 1, & x>3 \end{cases}$ Find c and also the p. d. f.

[Ans: c = 1/16, p. d. f., $f(x)=\begin{cases} 0, & x\le 1 \\ \frac{1}{4}(x-1)^3, & 1\le x\le 3 \\ 0, & x>3 \end{cases}$

12. Suppose that the error in the reaction temperature, in ^{0}C, for a controlled laboratory experiment is a Random variable X having the p. d. f. $f(x)=\begin{cases} \frac{x^2}{3}, & -1<x<2 \\ 0, & other\ wise \end{cases}$

Find (i) F(x) and (ii) Use it to evaluate $P(0<X\le 1)$

Answer: $F(x)=\begin{cases} 0 & x\le -1 \\ \frac{x^3+1}{9}, & -1<x<2 \\ 1, & x\ge 2 \end{cases}$ $, P[0<x\le 1]=\frac{1}{9}$

13. If the p. d. f. of a Random variable X is given by $f(x)=\begin{cases} 2kxe^{-x^2}, & for\ x>0 \\ 0, & for\ x\le 0 \end{cases}$

Find (a) the value of k and (b) distribution function for X.

[Ans: k = 1, $F(x)=\begin{cases} 1-e^{-x^2}, & for\ x\ge 0 \\ 0, & other\ wise \end{cases}$

14. Find the c. d. f. of the random variable whose p. d. f. is given by $f(x)=\begin{cases} \frac{x}{2}, & for\ 0<x<1 \\ \frac{1}{2}, & for 1<x\le 2 \\ \frac{3-x}{2}, & for\ 2<x\le 3 \\ 0, & other\ wise \end{cases}$

$$F(x)=\begin{cases}0, & x\le 0\\ \dfrac{x^2}{4}, & 0<x\le 1\\ \dfrac{2x-1}{4}, & 1<x\le 2\\ \dfrac{6x-x^2-5}{4}, & 2<x\le 3\\ 1, & x>3\end{cases}$$

15. A random variable X has the density function (i) $f(x)=\dfrac{k}{1+x^2}, -\infty<x<\infty$. Determine k & hence evaluate: (i) $P(x\ge 0)$ (ii) $P(0<x<1)$ [Ans: k = 1/π, (i) ½ (ii) ¼]

<u>Discrete Probability Distributions</u>:

Repeated trials:

A random experiment with only two possible outcomes categorized as **success** and **failure** is called a Bernoulli trial where the probability of success 'p' is same for each trial.

If a trial is repeated 'n' times and if 'p' is the probability of a success and 'q' that of a failure, then the probability of 'x' successes and (n – x) failures is given by $p^x q^{n-x}$. But these 'n' successes and (n – x) failures can occur in any of the ${}^nC_x$ ways in each of which the probability is same. Thus the probability of 'x' successes is ${}^nC_x p^x q^{n-x}$.

There are two Probability Distributions. They are (1) Binomial distribution (2) Poisson distribution

1. **<u>Binomial Distribution</u>**: (James Bernoulli)

It is concerned with trials of a repetition nature in which only the occurrence or non – occurrence, success or failure, acceptance or rejection, yes or no of a particular event is of interest.

If we perform a series of independent trials such that for each trial p is the probability of a success and q that of a failure. Then the probability of x successes in a series of n trials is given by

$p(X=x)=P(x)={}^nC_x p^x q^{n-x};\ \ x=0,1,2,3,\ldots\ldots,n$

p(X) is a Binomial Probability distribution.

We form the following probability distribution of [x, P(x)], where x = 0, 1, 2, ……, n

x	0	1	2			r			n
p(x)	q^n	${}^nC_1 pq^{n-1}$	${}^nC_2 p^2 q^{n-2}$			${}^nC_r p^r q^{n-r}$			p^n

p(x) for different values of x = 0, 1, 2, ……., n are the successive terms in the binomial expansion of $(q + p)^n$. There fore this distribution is called the Binomial Distribution.

Now $\sum p(x) = q^n + {}^nC_1 pq^{n-1} + {}^nC_2 p^2 q^{n-2} + ...+ p^n = (q+p)^n = 1^n = 1$

Hence p(x) is a probability function. n & p are the parameters of distribution.

Mean (Expectation) & Variance of the Binomial distribution: (IMP)

Mean:

$$E(X) = \mu = \sum_{x=0}^{n} xp(x) = \sum_{x=0}^{n} x.{}^nC_x p^x q^{n-x} = 0 + 1.{}^nC_1 pq^{n-1} + 2.{}^nC_2 p^2 q^{n-2} + ...+ {}^nC_n p^n q^0$$

$$= npq^{n-1} + n(n-1)p^2 q^{n-2} + ...+ np^n = np\left(q^{n-1} + (n-1)pq^{n-2} + ...+ p^{n-1}\right)$$

$$= np(q+p)^{n-1} = np.1^{n-1} = np$$

Hence $E(X) = \mu = np$

Variance:

$$\sigma^2 = E(X^2) - \{E(X)\}^2 = \sum_{x=0}^{n} x^2 p(x) - (np)^2 \left[Now\ x^2 = x(x-1) + x \ \&\ p(x) = {}^nC_x p^x q^{n-x}\right]$$

$$= \sum_{x=0}^{n} [x(x-1) + x]\left({}^nC_x p^x q^{n-x}\right) - n^2 p^2 = \sum_{x=0}^{n} x(x-1).\ {}^nC_x p^x q^{n-x} + \sum_{x=0}^{n} x.\ {}^nC_x p^x q^{n-x} - n^2 p^2$$

$$= 0 + 0 + 2.1.\ {}^nC_2 p^2 q^{n-2} + 3.2.\ {}^nC_3 p^3 q^{n-3} + ...+ n(n-1).\ {}^nC_n p^n q^0 + np - n^2 p^2$$

$$= n(n-1)p^2 q^{n-2} + 3.2.\frac{n(n-1)(n-2)}{3.2.1} p^3 q^{n-3} + ...+ n(n-1)p^n + np - n^2 p^2$$

$$= n(n-1)p^2\left[q^{n-2} + (n-2)pq^{n-3} + ...+ p^{n-2}\right] + np - n^2 p^2 \quad = n(n-1)p^2 (q+p)^{n-2} + np - n^2 p^2$$

$$= n^2 p^2 - np^2 + np - n^2 p^2 = np(1-p) = n\,p\,q$$

$$\therefore \sigma^2 = n\,p\,q \text{ Or } \sigma = \sqrt{n\,p\,q}$$

Binomial Frequency Distribution:

Fitting Binomial distribution: If 'n' independent trials constitute one experiment and this experiment be repeated 'N' times, then the frequency of 'x' success is

$N\ p(x) = N \times \left({}^nC_x p^x q^{n-x}\right)$. The possible number of successes together with these expected frequencies constitutes binomial frequency distribution.

Application of binomial distribution:

This distribution is applied to problems concerning;

(i) Number of defectives in a sample from production line

(ii) Estimation of reliability of system

(iii) Number of rounds fired from a gun hitting a target.

(iv) Radar detection.

Problems:

1.The probability that a pen manufacture by a company will be defective is $1/10$. If 12 such pens are manufactured, find the probability that (a) exactly two will be defective

(b) at least two will be defective (c) none will be defective. (VTU 2003)

Answer: X: Number of defective pen, n = 12, p = probability of a defective pen = $1/10$ = 0.1, q = 1 - p = 1- 0.1 = 0.9

$\therefore P(X = x) = p(x) = {}^{n}C_x p^x q^{n-x} = {}^{12}C_x (0.1)^x (0.9)^{12-x}$

(a) $P\,[exactly\ 2\ will\ be\ defective] = P(X = 2) = {}^{12}C_2 (0.1)^2 (0.9)^{12-2} = 0.2301$

(b) $P[at\ least\ two\ will\ be\ defective] = P(X \geq 2) = 1 - P(X < 2) = 1 - [p(0) + p(1)]$ $(x = 0\ or\ 1)$

$= 1 - [{}^{12}C_0 (0.1)^0 (0.9)^{12-0} + {}^{12}C_1 (0.1)^1 (0.9)^{12-1}] = 1 - [(0.9)^{12} + 12 \times 0.1 \times (0.9)^{11}] = 1 - 0.6590 = 0.3410$

(c) $P[that\ none\ will\ be\ defective] = P(X = 0) = {}^{12}C_0 (0.1)^0 (0.9)^{12} = 0.2824$

2. In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts?

Solution: Mean number of defective = 2 = n p = 20 p $\Rightarrow$ Probability of a defective part $= p = 2/20 = 0.1$

$\Rightarrow$ probability of non defective parts = $q = 1 - p = 0.9$, n = 20, p = 0.1 & q = 0.9

Hence $p(x) = {}^{n}C_x p^x q^{n-x} = {}^{20}C_x (0.1)^x (0.9)^{20-x}$

$\therefore$ Probability of at least 3 defectives in a sample of 20 is

$P(X \geq 3) = 1 - P(X < 3) = 1 - [p(0) + p(1) + p(2)]$

$$= 1 - \left[{}^{20}C_0 (0.1)^0 (0.9)^{20-0} + {}^{20}C_1 (0.1)^1 (0.9)^{20-1} + {}^{20}C_2 (0.1)^2 (0.9)^{20-2}\right]$$

$$= 1 - [0.9^{20} + 20 \times 0.1 \times (0.9)^{19} + 10 \times 19 \times (0.1)^2 \times (0.9)^{18}]$$

$$= 1 - (0.9)^{18} \times 4.51 = 0.3231$$

There in 1000 samples, the expected number of sample having at least 3 defectives is

$=1000\times0.3231=323$

3) Fit a binomial distribution for the following data & find expected frequencies.

X = x	0	1	2	3	4	Total
frequency	30	62	46	10	2	150 = N

X = x	f	f.x
0	30	0
1	62	62
2	46	92
3	10	30
4	2	8
Total	$\sum f = N = 150$	$\sum f\,x = 192$

$$p(x) = {}^{n}C_x\, p^x q^{n-x} \quad x = 0,\ 1,\ 2,\ 3,\ 4\ \ \&\ \ p+q=1,\ \ \bar{x} = \frac{\sum f\,x}{N} = \frac{192}{150} = 1.28 = \mu$$

$$\therefore np = 1.28 \Rightarrow p = \frac{1.28}{n} = \frac{1.28}{4} = 0.32\ \&\ \ q = 1-p = 1-0.32 = 0.68$$

Hence $p(x) = {}^{4}C_x (0.32)^x (0.68)^{4-x}$, x = 0, 1, 2, 3, 4, Expected frequency = N × p(x)

$$p(0) = {}^{4}C_0 (0.32)^0 (0.68)^{4-0} = (0.68)^4 = 0.2138,\ E(X=0) = N\times p(0) = 150\times0.2138 = 32.07 \approx 32$$

$$p(1) = {}^{4}C_1 (0.32)^1 (0.68)^{4-1} = 4\times(0.32)(0.68)^3 = 0.4025,\ E(X=1) = N\times p(1) = 150\times0.4025 = 60$$

$$p(2) = {}^{4}C_2 (0.32)^2 (0.68)^{4-2} = 6\times(0.32)^2(0.68)^2 = 0.2841,\ E(X=2) = N\times p(2) = 150\times0.2841 = 43$$

$$p(3) = {}^{4}C_3 (0.32)^3 (0.68)^{4-3} = 4\times(0.32)^3(0.68) = 0.0891,\ E(X=3) = N\times p(3) = 150\times0.0891 = 13$$

$$p(4) = {}^{4}C_4 (0.32)^4 (0.68)^{4-4} = 1\times(0.32)^4 = 0.0105,\ E(X=4) = N\times p(4) = 150\times0.0105 = 2$$

Exercises I: 1) Fit a binomial distribution to the following frequency distribution.

x	0	1	2	3	4	5
f	2	14	20	34	22	8

2. Out of 800 families with 5 children each, how many would you expect to have (a) 3 boys (b) 5 girls (c) either 2 or 3 boys? Assume equal probabilities for boys & girls.
3. If 10 percent of the rivets produces by a machine are defective, find the probability that out of 12 rivets chosen at random (i) exactly 2 will be defective (ii) at least 2 will be defective (iii) none will be defective.

Poisson Distribution:

Poison distribution is limiting case of Binomial distribution. It can be derived as a limiting case of B.D. by making n very large and 'p' very small, keeping np fixed (= m say).

The probability of 'x' success out of n trials in a binomial distribution is

$$p(x) = {}^nC_x p^x q^{n-x}, \text{ where } q = 1-p$$

$$= \frac{n(n-1)(n-2).......(n-x+1)}{x!} P^x q^{n-x}$$

As n tends to infinity (∞) p tends to 0 and taking np as a fixed quantity say np = m, $\Rightarrow$ $p = m/n.$ we get,

$$p(x) = \frac{m(m-m/n)(m-2m/n)(m-3m/n).......(m-xm/n)}{x!}(1-m/n)^{n-x}$$

$$= \frac{m^x}{x!} \underset{n\to\infty}{Lim} \frac{(1-m/n)^n}{(1-m/n)^x} = \frac{m^x}{x!} e^{-m}, \text{ where x = 0, 1, 2, 3, ...}$$

$$\left(\because \underset{n\to\infty}{Lim}(1-m/n)^n = e^{-m}, as \underset{k\to 0}{Lim}(1+k)^{1/k} = e \;\&\; \underset{n\to\infty}{Lim}(1-m/n)^x = 1\right)$$

'm' is the parameter, the mean number of occurrences.

Note: 1.p(x) is the probability of exactly 'x' occurrences.

2. p(x) is called Poisson probability function and 'x' is a Poisson variate.

The distribution of probability for x = 0, 1, 2, ……. is as follows:

x	0	1	2	3	……	…...
$p(x)$		$\frac{m}{1!}e^{-m}$	$\frac{m^2}{2!}e^{-m}$	$\frac{m^3}{3!}e^{-m}$	……	……

$\therefore$ we have $P(x) \geq 0 \;\&\; \sum_{x=0}^{\infty} P(x) = e^{-m} + \frac{me^{-m}}{1!} + \frac{m^2}{2!}e^{-m} + \frac{m^2 e^{-m}}{2!} + ...$

$$= e^{-m}\left[1+\frac{m}{1!}+\frac{m^2}{2!}+\frac{m^3}{3!}+.......\right]= e^{-m}.e^{m} = 1$$

Hence P(x) is a probability mass function.

Mean and Variance of Poison Distribution:

Mean = $\mu = \sum_{x=0}^{\infty} xP(x) = \sum_{x=0}^{\infty} x.\frac{m^x e^{-m}}{x!} = \sum_{x=1}^{\infty}\frac{m^x e^{-m}}{(x-1)!} = me^{-m}\sum_{x=1}^{\infty}\frac{m^{x-1}}{(x-1)!}$

$$= me^{-m}\left[1+\frac{m}{1!}+\frac{m^2}{3!}+.....\right]= m.e^{-m}.e^{m} = m \quad \therefore mean\ (\mu) = m$$

Variance (V) = E(X^2) – (E(X))^2 $= \sum_{x=0}^{\infty} x^2 p(x) - m^2 = \sum_{x=0}^{\infty}[x(x-1)+x]\frac{m^x e^{-m}}{x!} - m^2$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{m^x e^{-m}}{x!} + \sum_{x=0}^{\infty} x\frac{m^x e^{-m}}{x!} - m^2 = \sum_{x=2}^{\infty}\frac{m^x e^{-m}}{(x-2)!} + \sum_{x=0}^{\infty} xp(x) - m^2$$

$$= m^2 e^{-m}\sum_{x=2}^{\infty}\frac{m^{x-2}}{(x-2)!} + m - m^2 = m^2 e^{-m}\left[1+\frac{m}{1!}+\frac{m^2}{2!}+........\right]+m-m^2$$

$$= m^2 e^{-m} e^{m} + m - m^2 = m^2 + m - m^2 = m$$

Hence $Variance\ (\sigma^2) = m$ & $SD\ (\sigma) = \sqrt{m}$

Problems: 1) A company receives three complaints per day on average. What is the probability of receiving more than one complaint on a particular day?

Solution: Here m = 3, $\therefore\ p(x) = \frac{m^x e^{-m}}{x!} = \frac{3^x e^{-3}}{x!},\quad x = 0, 1, 2, 3.........$

Therefore probability of receiving more than one complaint $= P(X > 1) = P(2) + P(3) + ...$

$$= 1 - P(X \le 1) = 1 - [P(0) + P(1)] = 1 - \left[\frac{3^0 e^{-3}}{0!} + \frac{3^1 e^{-3}}{1!}\right] = 1 - [0.0498 + 0.1494] = 0.8008$$

2) If the probability of a bad reaction from a certain injection is 0.001, determine the chance that out of 2000 individuals more than two will get a bad reaction. (VTU 2003).

Solution: It follows a Poisson distribution as the probability of occurrence is very small.

Mean = m = np = 2000×0.001=2, Hence $P(x) = \frac{2^x e^{-2}}{x!}$

Therefore probability more than two will get a bad reaction is $= P(X > 2) = 1 - P(X \le 2)$

$$= 1 - [P(0) + P(1) + P(2)] = 1 - \left[e^{-2} + \frac{2e^{-2}}{1} + \frac{2^2 e^{-2}}{2!} \right] = 1 - \frac{1}{e^2}(1 + 2 + 2) = 1 - \frac{5}{e^2} = 0.3233$$

3) In a certain factory turning out razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10, use Poisson distribution to calculate the approximate number of packets containing no, one defective & 2 defective blades respectively in a consignment of 10, 000 packets. (VTU 2004)

Solution: Here n = 10, p = 0.002, N = 10, 000, Then m = np = 10 (0.002) = 0.02

Therefore $P(x) = \frac{m^x e^{-m}}{x!} = \frac{(0.02)^x e^{-0.02}}{x!}$

(i) Probability of no defective blade (i.e. X = 0) = $P(0) = \frac{(0.02)^0 e^{-0.02}}{0!} = 0.9802$

$\therefore$ Number of packets containing no defective blade is $N \times p(0) = 10000 \times 0.9802 = 9802$

(ii) Number of packets containing one defective blade is $N \times P(1) = 10000 \times \frac{(0.02)^1 e^{-0.02}}{1!} = 196.$

(iii) Number of packets containing two defective blade is $N \times P(2) = 10000 \times \frac{(0.02)^2 e^{-0.02}}{2!} = 1.96 \approx 2$

4) Fit a Poisson distribution to the following data: (VTU 2004)

x:	0	1	2	3	4
f:	122	60	15	2	1

Find the corresponding theoretical estimation for f.

Solution:

x	F	f.x
0	122	0
1	60	60
2	15	30
3	2	6
4	1	4
Total	N =200	$\sum fx = 100$

$\bar{x} = \frac{\sum fx}{N} = \frac{100}{200} = 0.5 = m,$ the mean of Poisson distribution. Hence

$p(x) = \frac{m^x e^{-m}}{x!} = \frac{(0.5)^x e^{0.5}}{x!},\ x = 0, 1, 2, 3, 4.$ Hence the expected frequency for 'x' successes is

$E_x = N \times P(x) = \frac{200 \times (0.5)^x e^{-0.5}}{x!}$, where x = 0, 1, 2, 3, 4. Putting x = 0, 1, 2, 3 , 4 we get

$E_0 = N \times P(0) = \frac{200 \times (0.5)^0 e^{-0.5}}{0!} = 121,\ \ E_1 = N \times P(1) = \frac{200 \times (0.5)^1 e^{-0.5}}{1!} = 61,$

$E_2 = N \times P(2) = \frac{200 \times (0.5)^2 e^{-0.5}}{2!} = 15,\ \ E_3 = N \times P(3) = \frac{200 \times (0.5)^3 e^{-0.5}}{3!} = 3,$

$E_4 = N \times P(4) = \frac{200 \times (0.5)^4 e^{-0.5}}{4!} = 0,$ Hence the theoretical frequencies are

x:	0	1	2	3	4
f:	121	61	15	3	0

Normal Distribution

The continuous probability distribution having the probability density function f(x) is given by

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2 / 2\sigma^2}$, where $-\infty < x < \infty, \quad -\infty < \mu < \infty$ and $\sigma > 0,$ is known as the normal distribution. μ & σ are the parameters.

Now clearly $f(x) \geq 0$ & $\int_{-\infty}^{\infty} f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

Put $t=\frac{x-\mu}{\sqrt{2}\sigma}$ or $x=\mu+\sqrt{2}\sigma\, t \Rightarrow dx=\sqrt{2}\ \sigma\, dt,$ Limit is from $-\infty$ *to* ∞

Hence $\int_{-\infty}^{\infty} f(x)dx=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2}\sqrt{2}\ \sigma\ dt=\frac{1}{\sqrt{\pi}}2\int_{-\infty}^{\infty} e^{-t^2}dt \quad \left(\because e^{-t^2} \text{ is an even function}\right)$

$=\frac{2}{\sqrt{\pi}}\times\frac{\sqrt{\pi}}{2}=1.$ Thus f(x) is a p. d. f.

<u>Mean and SD of the normal distribution</u>:

$$Mean\,(\mu)=\int_{-\infty}^{\infty} xf(x)dx=\int_{-\infty}^{\infty} x.\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} x.e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$$

$put\ t=\frac{x-\mu}{\sqrt{2}\sigma}\ or\ x=\mu+\sqrt{2}\sigma\, t \Rightarrow dx=\sqrt{2}\sigma\ dt$

$$=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(\mu+\sqrt{2}\sigma t)e^{-t^2}\sqrt{2}\sigma\ dt \quad (\lim\ of\ t\ is\ -\infty\ to\ \infty)$$

$$=\frac{\mu}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}dt+\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} te^{-t^2}dt$$

$$=\frac{2\mu}{\sqrt{\pi}}\int_{0}^{\infty} e^{-t^2}dt+\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} te^{-t^2}dt \quad \left(\because as\ e^{-t^2} is\ an\ even\ function \int_{-\infty}^{\infty} e^{-t^2}dt=2\int_{0}^{\infty} e^{-t^2}dt\right)$$

$$=\frac{2\mu}{\sqrt{\pi}}\times\frac{\sqrt{\pi}}{2}+\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\times 0 \quad \left(\because as\ te^{-t^2} is\ an\ odd\ function \int_{-\infty}^{\infty} te^{-t^2}dt=0\right)$$

$$=\mu$$

Hence mean of the normal distribution is equal to the mean of the given distribution.

Variance = $\sigma^2=\int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)^2.e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$

$put\ t=\frac{x-\mu}{\sqrt{2}\sigma}\ or\ x=\mu+\sqrt{2}\sigma\, t \Rightarrow dx=\sqrt{2}\sigma\ dt$

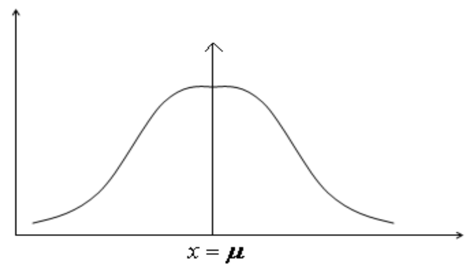
$$=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sigma^2 t^2.e^{-t^2}\sqrt{2}\sigma\ dt \ =\frac{2\sigma^2}{\sqrt{\pi}}2\int_{0}^{\infty} t^2.e^{-t^2}\ dt \quad \left(as\ t^2e^{-t^2} is\ even\right)$$

$=\frac{2\sigma^2}{\sqrt{\pi}}\int_{0}^{\infty} t.\left(2te^{-t^2}\right)dt$ (integration by parts we get)

$$= \frac{2\sigma^2}{\sqrt{\pi}} \left\{ \left[t\left(e^{-t^2}\right) \right]_0^{\infty} - \int_0^{\infty} -\left(e^{-t^2} dt\right) \right\} = \frac{2\sigma^2}{\sqrt{\pi}} \left\{ 0 + \int_0^{\infty} e^{-t^2} dt \right\} = \frac{2\sigma^2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} = \sigma^2$$

Hence variance = σ^2 & $S.D. = \sigma$

The line $x = \mu$ divides the total area under the curve which is equal to 1 into two equal parts. The area to the right as well as to the left of the line $x = \mu$ is 0.5



Properties of the normal Distribution:

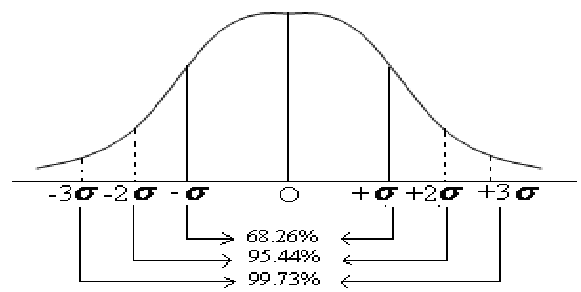
1. The normal curve is bell shaped and is symmetric about its mean.
2. It is unimodal with ordinates decreasing rapidly on both sides of the mean.
3. Mean = Median = mode
4. The maximum ordinate is $\frac{1}{\sigma\sqrt{2\pi}}$, found by putting $x = \mu$ in $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2 / 2\sigma^2}$
5. **The area property**: Total area = 1

   (i) The area under the normal curve between the ordinates $x = \mu - \sigma$ & $x = \mu + \sigma$ is $0.6826 \approx 68\%$ *nearly.* Thus approximately $2/3$ of the values lie within these limits.

   i.e. $P(\mu - \sigma \le X \le \mu + \sigma) = 0.6826$

   (ii) $P(\mu - 2\sigma \le X \le \mu + 2\sigma) = 0.9544 \approx 95.44\%$

   (iii) $P(\mu - 3\sigma \le X \le \mu + 3\sigma) = 0.9973 \approx 99.73\%$ and so on.

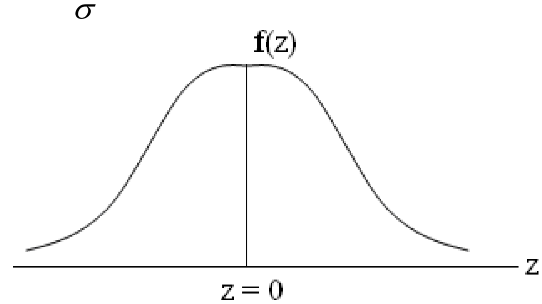


<u>Standard Normal Distribution</u>: Normal distribution with mean = 0 and S.D. = 1 is called a standard normal distribution. It is denoted by Z. Its p. d. f. is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad -\infty < z < \infty$$

Let 'X' be a normal Variable with mean 'μ' & S.D. σ then $z = \frac{X - \mu}{\sigma}$ is a standard normal variable with mean 0 and S.D. = 1

It is symmetrical about the line z = 0



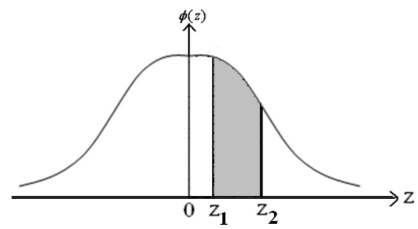
Distribution Function F(x) (c. d. f.):

$$F(x) = \int_{-\infty}^{x} f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$P(a < X < b) = \int_{a}^{b} f(x)dx = P(a \le X \le b)$$

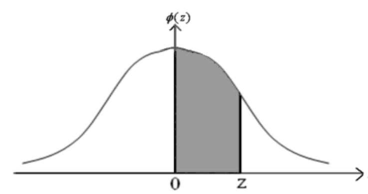
Let $z_1 = \frac{a-\mu}{\sigma}$ & $z_2 = \frac{b-\mu}{\sigma}$ be the values of z corresponding to X = a & X = b, then

$$P(a < X < b) = P(z_1 \le z \le z_2) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-z^2/2} dz$$



i.e. area under the normal curve between z = z_1 & z = z_2 as shown in the figure.

If $z_1 = 0$ & $z_2 = z$, we have $\phi(z) = \frac{1}{\sqrt{2\pi}} \int_{0}^{z} e^{-z^2/2} dz$

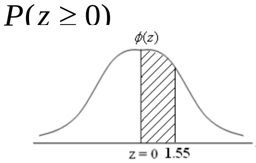


i.e. area under the normal curve between 0 & z as shown in the figure.

Note: It is not possible to integrate $\int_{a}^{b} e^{-x^2} dx$, (We can use numerical integration). The results are available in special table called **normal distribution table**.

Some important results:

1. $P(-\infty \le z \le \infty) = 1$ 2. $P(-\infty \le z \le 0) = 1/2 = P(z \le 0)$ 3. $P(0 \le z \le \infty) = 1/2 = P(z \ge 0)$

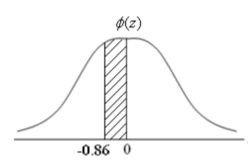


4. $P(-\infty \le z \le z_1) = P(-\infty \le z \le 0) + P(0 \le z \le z_1) = 0.5 + \phi(z_1) = P(z \le z_1)$

5. $P(z \ge z_2) = P(z \ge 0) - P(0 \le z \le z_2) = 0.5 - \phi(z_2)$

Example: 1. Find the area under the standard normal curve between z = 0 & z = 1.55.

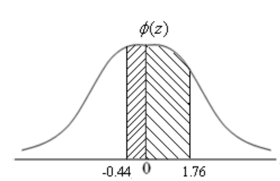
Solution: Area = $\frac{1}{\sqrt{2\pi}} \int_{0}^{1.55} e^{-z^2/2} dz = \phi(1.55) = 0.4394$



2.Find the area under the standard normal curve between z = -0.86 & z = 0.

Solution: Area = $\frac{1}{\sqrt{2\pi}} \int_{-0.86}^{0} e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} \int_{0}^{0.86} e^{-z^2/2} dz = \phi(0.86) = 0.3051$

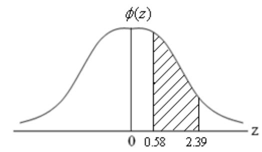
3. Find the area under the standard normal curve between z = -0.44 & z = 1.76



Solution: Area = $\frac{1}{\sqrt{2\pi}}\int_{-0.44}^{1.76} e^{-z^2/2}dz = \frac{1}{\sqrt{2\pi}}\int_{-0.44}^{0} e^{-z^2/2}dz + \frac{1}{\sqrt{2\pi}}\int_{0}^{1.76} e^{-z^2/2}dz$

$$= \phi(0.44) + \phi(1.76) = 0.1700 + 0.4608 = 0.6308$$

4. Find the area under the standard normal curve between z = 0.58 & z = 2.39



Solution: Area = $\frac{1}{\sqrt{2\pi}}\int_{0.58}^{2.39} e^{-z^2/2}dz = \frac{1}{\sqrt{2\pi}}\int_{0}^{2.39} e^{-z^2/2}dz - \frac{1}{\sqrt{2\pi}}\int_{0}^{0.58} e^{-z^2/2}dz$

$$= \phi(2.39) - \phi(0.58) = 0.4916 - 0.2190 = 0.2726$$

5. Evaluate the following: i) $P(z \geq 0.85)$ ii) $P(-1.64 \leq z \leq -0.88)$ iii) $P(z \leq -2.43)$ iv) $P(|z| \leq 1.94)$

v) $P(z \geq -1.76)$

Answer: i) $P(z \geq 0.85) = P(z \geq 0) - P(z \leq 0.85) = 0.5 - \phi(0.85) = 1.1977$

ii) $P(-1.64 \leq z \leq -0.88) = \phi(1.64) - \phi(0.88) = 0.1389$

iii) $P(z \leq -2.43) = P(z \geq 0) - P(z \leq 2.43) = 0.5 - \phi(2.43) = 0.0075$

iv) $P(|z| \leq 1.94) = P(-1.94 \leq z \leq 1.94) = 2P(0 \leq z \leq 1.94) = 2\phi(1.94) = 0.9476$

v) $P(z \geq -1.76) = 0.5 + P(z \leq 1.76) = 0.5 + \phi(1.76) = 0.5 + 0.4608 = 0.9608$

Problems: (1) Suppose temperature in may follows N(38, 3^2), Find the probability of temperature is,

(i) More than 40 degrees (x > 40) (ii) less than 35 degrees (x < 35) (iii) between 32 & 36 degrees (32 < X < 36)

Solution: Let random variable X denotes the variation in temperature, $\mu = 38^0$, $\sigma = 3^0$

(i) $z = \frac{x-\mu}{\sigma} = \frac{x-38}{3}$, $P(X > 40) = P\left(z > \frac{x-\mu}{\sigma}\right) = P\left(z > \frac{40-38}{3}\right) = P(z > 0.67)$

$$= P(z > 0) - P(z \leq 0.67) = 0.5 - \phi(0.67) = 0.5 - 0.2486 = 0.2514$$

(ii) $P(X < 35) = P\left(z < \frac{35-38}{3}\right) = P(z < -1) = P(z > 1) = P(z > 0) - P(z < 1) = 0.5 - \phi(1) = 0.1587$

(iii) $P(32 \leq X \leq 36) = P\left(\frac{32-38}{3} \leq z \leq \frac{36-38}{3}\right) = P(-2 \leq z \leq -0.67) = \phi(2) - \phi(0.67) = 0.2286$

(2) In a test on 2500 electric bulbs, it was found that the life of a particular make, was normally distributed with an average life of 2000 hours and S.D. of 60 hours. Estimate the number of bulbs likely to burn for (a) more than 2100 hours (b) less than 1950 hours (c) more than 1900 hours & but less than 2100 hours. (VTU 2004)

Solution: X: life of an electric bulb measured in hours. Then $\mu = 2000$, $\sigma = 60$. Therefore

$z = \dfrac{x-\mu}{\sigma} = \dfrac{x-2000}{60}$,

(a) $P(X > 2100) = P\left(z > \dfrac{2100-2000}{60}\right) = P(z > 1.67 = 0.5 - \phi(1.67) = 0.5 - 0.4525 = 0.0475$

Thus the number of bulbs expected to burn for more than 2100 hours= $0.0475 \times 2500 \approx 119$ bulbs

(b) P(X < 1950) $= P\left(z < \dfrac{1950-2000}{60}\right) = P(z < -0.83) = P(z \geq 0.83) = 0.5 - \phi(0.83) = 0.2023$

Thus the number of bulbs expected to burn for less than 1950 hours = $0.2023 \times 2500 = \approx 508$ bulbs

(iii) $P(1900 < X < 2100) = P\left(\dfrac{1900-2000}{60} < z < \dfrac{2100-2000}{60}\right) = P(-1.67 < z < 1.67)$

$= 2\phi(1.67) = 0.905$

Thus the number of bulbs expected to burn for more than 1900 hours & less than 2100 hours

$= 0.905 \times 2500 = 2263$ bulbs.

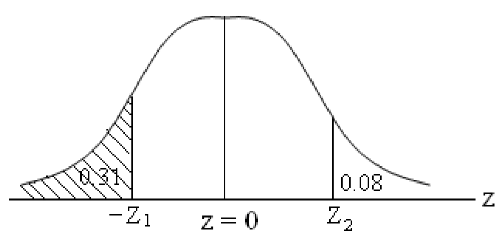
(3) The mean & SD of the marks obtained by 1000 students in an examination are respectively 70 & 5. Assuming normality of the distribution, find the approximate number of students whose marks will be

(i) Less than 65 (ii) more than 75 (iii) between 65 & 75. (Answer: 159, 159, 683)

(4) In a normal distribution 31% of the items are under 45 and 8% of the items are over 64. Find the mean & SD of the distribution.

Solution: Let $\mu \,\&\, \sigma$ be the mean & SD of the normal distribution.

Given that $P(X < 45) = 0.31$ & $P(X > 64) = 0.08$



We have $z = \dfrac{x-\mu}{\sigma}$, When x = 45, $z = \dfrac{45-\mu}{\sigma} = z_1$ (say)

When x = 64, $z = \dfrac{64-\mu}{\sigma} = z_2$ (say)

$\therefore P(z < z_1) = 0.31$ & $P(z > z_2) = 0.08$. i.e. $0.5 + \phi(z_1) = 0.31$ & $0.5 - \phi(z_2) = 0.08$

$\Rightarrow \phi(z_1) = -0.19$ & $\phi(z_2) = 0.42$. But from the normal table 0.1915 (≈0.19) = $\phi(0.5)$ &

0.4192 (≈0.42) = $\phi(1.4)$. $\Rightarrow z_1 = -0.5$ & $z_2 = 1.4$.

Hence $\frac{45-\mu}{\sigma} = -0.5$ & $\frac{64-\mu}{\sigma} = 1.4 \Rightarrow \mu - 0.5\sigma = 45$ & $\mu + 1.4\sigma = 64$. By solving we get

$\mu = 50$ & $\sigma = 10 \Rightarrow$mean = 50 & S. D = 10.

Fitting Normal distribution:

(1) Obtain the equation of the normal probability curve that may be fitted to the following data:

X = x	6	7	8	9	10	11	12
Frequency	3	6	9	13	8	5	4

Solution: The equation of best fitting normal curve is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Therefore we have to compute mean (μ) & S.D. (σ) for the given frequency distribution.

x	f	$f.x$	$f.x^2$
6	3	18	108
7	6	42	294
8	9	72	576
9	13	117	1053
10	8	80	800
11	5	55	605
12	4	48	576
	$N = \sum f = 48$	$\sum fx = 432$	$\sum fx^2 = 4012$

Hence $\mu = \frac{\sum fx}{\sum f} = \frac{432}{48} = 9$ & $\sigma^2 = \frac{\sum fx^2}{\sum f} - \mu^2 = \frac{4012}{48} - 9^2 = 2.5833$ Or $\sigma = \sqrt{2.5833} = 1.607$

Thus the required equation of the normal probability curve is

$$f(x) = \frac{1}{1.607\sqrt{2\pi}} e^{-\frac{(x-9)^2}{5.167}} = 0.2483 e^{-0.1935(x-9)^2}$$

(2) Fit a normal curve to the following distribution

x:	2	4	6	8	10	
f:	1	4	6	4	1	(VTU 2001)

Solution:

x	f	$f.x$	$f.x^2$
2	1	2	4
4	4	16	64
6	6	36	216
8	4	32	256
10	1	10	100
	$\sum f = 16$		$\sum fx^2 = 640$

$$\therefore \mu = \frac{\sum fx}{\sum f} = \frac{96}{16} = 6. \ \& \ S.D. = \sigma = \sqrt{\frac{\sum fx^2}{\sum f} - \mu^2} = \sqrt{\frac{640}{16} - 6^2} = \sqrt{40-36} = 2$$

Therefore the equation of normal curve is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-6)^2}{8}} = 0.1995 e^{-0.125(x-6)^2}$$

Exponential distribution: The continuous probability distribution having the p.d.f. f(x) given by $f(x) = \begin{cases} \alpha e^{-\alpha x} & for\ x > 0 \\ 0 & other\ wise \end{cases}$ where $\alpha > 0$ is known as the exponential distribution. Then clearly

$$f(x) > 0 \ \& \ \int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} \alpha e^{-\alpha x} dx = \left[\alpha \frac{e^{-\alpha x}}{-\alpha} \right]_0^{\infty} = -(0-1) = 1 \Rightarrow \text{f(x) is a p.d.f.}$$

Mean and S.D. of the exponential distribution:

Mean $(\mu) = \int_{-\infty}^{\infty} x f(x)dx = \int_0^{\infty} x.\alpha e^{-\alpha x} dx = \alpha \int_0^{\infty} x e^{-\alpha x} dx,$ By Applying Bernoulli's rule of integration by parts we get, $\mu = \alpha \left[x.\frac{e^{-\alpha x}}{-\alpha} - 1.\frac{e^{-\alpha x}}{\alpha^2} \right]_0^{\infty} = \alpha \left[0 - \frac{1}{\alpha^2}(0-1) \right] = \frac{1}{\alpha} \quad \therefore \mu = \frac{1}{\alpha}$

Variance:

Variance = $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \alpha \int_0^{\infty} (x-\mu)^2 e^{-\alpha x} dx,$ Applying Bernoulli's rule, we have

$$\sigma^2 = \alpha\left[(x-\mu)^2\frac{e^{-\alpha x}}{-\alpha} - 2(x-\mu)\frac{e^{-\alpha x}}{\alpha^2} + 2\frac{e^{-\alpha x}}{-\alpha^3}\right]_0^\infty$$

$$= \alpha\left[-\frac{1}{\alpha}(0-\mu^2) - \frac{2}{\alpha^2}(0+\mu) - \frac{2}{\alpha^3}(0-1)\right] = \alpha\left[\frac{\mu^2}{\alpha} - \frac{2\mu}{\alpha^2} + \frac{2}{\alpha^3}\right] \text{ , But } \mu = \frac{1}{\alpha}$$

$$= \alpha\left[\frac{1}{\alpha^3} - \frac{2}{\alpha^3} + \frac{2}{\alpha^3}\right] = \frac{1}{\alpha^2} \Rightarrow \sigma^2 = \frac{1}{\alpha^2} \text{ or } \sigma = \frac{1}{\alpha}$$

Thus for exponential distribution $mean\,(\mu) = \frac{1}{\alpha}$ & $S.D.\,(\sigma) = \frac{1}{\alpha}$

Problems:

(1) If ‘x’ is an exponential variate with mean 3 find (i) $P(X > 1)$ (ii) $P(X < 3)$

Solution: The p.d.f. of exponential distribution is given by $f(x) = \begin{cases} \alpha e^{-\alpha x} & for\ x > 0 \\ 0 & other\ wise \end{cases}$

Mean of exponential distribution is $\frac{1}{\alpha} = 3$ (given) $\Rightarrow \alpha = \frac{1}{3}$ $\therefore$ $f(x) = \begin{cases} \frac{1}{3}e^{-\frac{x}{3}} & for\ x > 0 \\ 0 & otherwise \end{cases}$

(i) $P(X > 1) = 1 - P(X \le 1) = 1 - \int_{-\infty}^{1} f(x)dx = 1 - \int_0^1 \frac{1}{3}e^{-\frac{x}{3}}dx = 1 - \frac{1}{3}\left[\frac{e^{-x/3}}{-1/3}\right]_0^1 = 1 + \left(e^{-\frac{1}{3}} - 1\right) = e^{-\frac{1}{3}} = 0.7165$

$\therefore P(X > 1) = 0.7165$

(ii) $P(X < 3) = \int_{-\infty}^{3} f(x)dx = \int_0^3 \frac{1}{3}e^{-\frac{x}{3}}dx = \frac{1}{3}\left[\frac{e^{-x/3}}{-1/3}\right]_0^3 = -\left(e^{-\frac{3}{3}} - 1\right) = 1 - e^{-1} = 0.6321$

$\therefore P(X < 3) = 0.6321$

(2) If x is an exponential variate with mean 5, evaluate (i) $P(0 < X < 1)$ (ii) $P(-\infty < X < 10)$

(iii) $P(X \le 0\ or\ X \ge 1)$

Solution: $f(x) = \alpha e^{-\alpha x}\quad x > 0$; Given mean = $5 = \frac{1}{\alpha} \Rightarrow \alpha = \frac{1}{5}$

Hence (i) $P(0 < X < 1) = \int_0^1 f(x)dx = \int_0^1 \frac{1}{5}e^{-\frac{x}{5}}dx \quad = \frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_0^1 = -(e^{-1/5} - 1)$

$$=1-e^{-0.2}=0.1813 \quad \therefore P(0<X<1)=0.1813$$

(ii) $P(-\infty<X<10)=\int_{-\infty}^{10} f(x)dx = \int_{-\infty}^{0} f(x)dx+\int_{0}^{10} f(x)dx=\int_{0}^{10}\frac{1}{5}e^{-x/5}dx = \frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_0^{10}$

$$=-\left(e^{-2}-1\right)=1-e^{-2}=0.8647 \qquad \therefore P(-\infty<X<10)=0.8647$$

(iii) $P(X\le 0 \text{ or } X\ge 1)=P(X\le 0)+P(X\ge 1)=\int_{-\infty}^{0} f(x)dx+\int_{1}^{\infty} f(x)dx$

$$=0+\int_{1}^{\infty}\frac{1}{5}e^{-x/5}dx \quad =\frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_1^{\infty} =\left(0-e^{-1/5}\right)=e^{-1/5}=0.8187$$

$\therefore$ $P(X\le 0 \text{ or } X\ge 1)=0.8187$

(3) The sales per day in a shop is exponential distributed with the average sale amounting to Rs. 100 and net profit is 8%. Find the probability that the net profit exceeds Rs. 30 on two consecutive days.

Solution: Let the random variable X denote the sale in the shop. Since x is an exponential variate its p.d.f. is given by $f(x)=\begin{cases}\alpha e^{-\alpha x} & for\ x>0\\ 0 & other\ wise\end{cases}$ Now mean $=\frac{1}{\alpha}=100 \Rightarrow \alpha=\frac{1}{100}=0.01$

Thus $f(x)=0.01e^{-0.01x},\ x>0$

Let A be the amount for which profit is 8% $\Rightarrow A.\frac{8}{100}=30$. $\therefore A=\frac{30\times 100}{8}=375$

Probability of profit exceeding Rs.30 is $P(profit>30)=1-P(profit\le 30)=1-P(Sales\le Rs.375)$

$$=1-P(X\le 375) \ =1-\int_{0}^{375} f(x)dx=1-\int_{0}^{375}0.01e^{-0.01x}dx$$

$$=1-0.01\left[\frac{e^{-0.01x}}{-0.01}\right]_0^{375} \ =1+\left(e^{-3.75}-1\right)=e^{-3.75}=0.0235$$

i.e. probability that profit exceeds Rs. 30 on a single day is 0.0235.

Hence probability that the profit exceeds Rs. 30 on two consecutive days = 0.0235×0.0235 = 0.00055

(4) In a certain town the duration of a shower (short fall of rain or slight rain) is exponentially distributed with mean 5 minutes. What is the probability that a shower will last for: (i) 10 minutes or more (ii) less than 10 minutes (iii) between 10 and 12 minutes.

Solution: X is exponential variate,

$$\therefore \quad f(x) = \begin{cases} \alpha e^{-\alpha x} & for\ x > 0 \\ 0 & otherwise \end{cases} \quad \text{\& mean } = \frac{1}{\alpha} = 5 \Rightarrow \alpha = \frac{1}{5}$$

Hence $f(x) = \frac{1}{5} e^{-\frac{x}{5}}, \quad x > 0$

(i) $P(X \geq 10) = \int_{10}^{\infty} f(x)dx = \frac{1}{5}\int_{10}^{\infty} e^{-\frac{x}{5}}dx = \frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_{10}^{\infty} = -(0 - e^{-2}) = e^{-2} = 0.1353$

(ii) $P(X < 10) = \frac{1}{5}\int_{0}^{10} e^{-x/5}dx = -\left[e^{-x/5}\right]_{0}^{10} = -(e^{-2} - 1) = 1 - e^{-2} = 0.8647$

(iii) $P(10 < X < 12) = \int_{10}^{12} f(x)dx = \frac{1}{5}\int_{10}^{12} e^{-\frac{x}{5}}dx$

$$= -\left[e^{-x/5}\right]_{10}^{12} = -(e^{-2.4} - e^{-2}) = e^{-2} - e^{-2.4} = 0.0446$$

Sampling Theory

Introduction

In statistics, a population is an entire set of objects or units of observation of one sort or another, while a sample is a subset (usually a proper subset) of a population, selected for particular study (usually because it is impractical to study the whole population). The numerical characteristics of a population are called parameters.

Generally the values of the parameters of interest remain unknown to the researcher; we calculate the "corresponding" numerical characteristics of the sample (known as statistics) and use these to estimate, or make inferences about, the unknown parameter values.

A standard notation is often used to keep straight the distinction between population and sample. The table below sets out some commonly used symbols.

Note that it's common to use a Greek letter to denote a parameter, and the corresponding Roman letter to denote the associated statistic.

Statistical Inference is a branch of Statistics which uses probability concepts to deal with uncertainty in decision making. There are a number of situations where in we come across problems involving decision making. For example, consider the problem of buying 1 kilogram of rice, when we visit the shop, we do not check each and every rice grains stored in a gunny bag; rather we put our hand inside the bag and collect a sample of rice grains. Then analysis takes place. Based on this, we decide to buy or not. Thus, the problem involves studying whole rice stored in a bag using only a sample of rice grains.

5.2 Hypothesis:

This topic considers two different classes of problems

1. Hypothesis testing – we test a statement about the population parameter from which the sample is drawn.
2. Estimation – A statistic obtained from the sample collected is used to estimate the population parameter.

First what is meant by hypothesis testing?

This means that testing of hypothetical statement about a parameter of population.

Conventional approach to testing:

The procedure involves the following:

1. First we set up a definite statement about the population parameter which we call it as null hypothesis, denoted by H_0. According to Professor R. A. Fisher, **Null Hypothesis is the statement which is tested for possible rejection under the assumption that it is true**.

Next we set up another hypothesis called alternate statement which is just opposite of null statement; denoted by H_1 which is just complimentary to the null hypothesis. Therefore, if we start with $H_0: \mu = \mu_0$ then alternate hypothesis may be considered as either one of the following statements;

$H_1 \quad \mu \neq \mu_0$, or $H_1: \mu > \mu_0$ or $H_1: \mu < \mu_0$.

As we are studying population parameter based on some sample study, one can not do the job with 100% accuracy since sample is drawn from the population and possible sample may not represent the whole population. Therefore, usually we conduct analysis at certain level of significance (lower than 100%. The possible choices include 99%, or 95% or 98% or 90%. **Usually we conduct analysis at 99% or 95% level of significance,** denoted by the symbol α . We test H_0 against H_1 at certain level of significance. The confidence with which a person rejects or accepts H_0 depends upon the significance level adopted. It is usually expressed in percentage forms such as 5% or 1% etc. Note that when α is set as 5%, then probability of rejecting null hypothesis when it is true is only 5%. It also means that when the hypothesis in question is accepted at 5% level of significance, then statistician runs the risk of taking wrong decisions, in the long run, is only 5%. The above is called II step of hypothesis testing.

Critical values or Fiducial limit values for a two tailed test:

Sl. No	Level of significance	Theoretical Value
1	1% $\alpha =$	2.58
2	2% $\alpha =$	2.33
3	5% $\alpha =$	1.96

<u>Critical values or Fiducial limit values for a single tailed test (right and test)</u>

Tabulated value	$\alpha =$ 1%	$\alpha =$ 5%	$\alpha =$ 10%
Right – tailed test	2.33	1.645	1.28
Left tailed test	-2.33	-1.645	-1.28

<u>Setting a test criterion</u>: The third step in hypothesis testing procedure is to construct a test criterion. This involves selecting an appropriate probability distribution for the particular test i.e. a proper probability distribution function to be chosen. Some of the distribution functions used are t, F, when the sample size is small (size lower than 30).

However, for large samples, normal distribution function is preferred. Next step is the computation of statistic using the sample items drawn from the population. Usually, samples are drawn from the population by a procedure called random, where in each and every data of the population has the same chance of being included in the sample. Then the computed value of the test criterion is compared with the tabular value; as long the calculated value is lower then or equal to tabulated value, we accept the null hypothesis, otherwise, we reject null hypothesis and accept the alternate hypothesis. Decisions are valid only at the particular level significance of level adopted.

During the course of analysis, there are two types of errors bound to occur. These are

(i) Type – I error and (ii) Type – II error.

Type – I error: This error usually occurs in a situation, when the null hypothesis is true, but we reject it i.e. rejection of a correct/true hypothesis constitute type I error.

Type – II error: Here, null hypothesis is actually false, but we accept it. Equivalently, accepting a hypothesis which is wrong results in a type – II error. The probability of committing a type – I error is denoted by α where

α = Probability of making type I error = Probability [Rejecting H_0 | H_0 is true]

On the other hand, type – II error is committed by not rejecting a hypothesis when it is false. The probability of committing this error is denoted by β. Note that

β = Probability of making type II error = Probability [Accepting H_1 | H_1 is false]

Critical region:

A region in a sample space S which amounts to Rejection of region. H_0 is termed as critical

One tailed test and two tailed test:

This depends upon the setting up of both null and alternative hypothesis.

A note on computed test criterion value:

1. When the sampling distribution is based on population of proportions/Means, then test criterion may be given as

$$z_{cal} = \frac{\text{Expected results - Observed results}}{\text{Standard error of the distribution}}$$

Application of standard error:

1. S.E. enables us to determine the probable limit within which the population parameter may be expected to lie. For example, the probable limits for population of proportion are given by $p \pm 3\sqrt{pqn}$. Here, p represents the chance of achieving a success in a single trial, q stands for the chance that there is a failure in the trial and n refers to the size of the sample.

   The magnitude of standard error gives an index of the precision of the parameter.

5. 3 Significance level:

The probability level below which leads to the hypothesis is known as the significance level. This probability is conventionally fixed at 0.05 or 0.01 i.e., 5% or 1%

Therefore rejecting hypothesis at 1% level of significance, implies that at 5% level of significance, there may be errors of either types (Type I or II) is 0.05.

TESTS OF SIGNIFICANCE AND CONFIDENCE INTERVALS

The process which helps us to decide about the acceptance or rejection of the hypothesis is called as the test of significance.

Suppose that we have a normal population with mean μ and S D as σ. If $\overline{x}$ is the sample mean of a random sample size (n), the quantity "t" defined by $t = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$

is called as the standard normal variate (SNV) whose $\overline{x} = 0$, $\sigma = 1$

From the table of the normal areas, we find that 95% of the area lies between $t = -1.96$ and $t = 1.96$

Further 5% level of significance is denoted by $t_{0.05}$, therefore, $-1.96 \leq \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} \leq 1.96$

$$\frac{-\sigma}{\sqrt{n}}(1.96) \leq \bar{x} - \mu \leq \frac{-\sigma}{\sqrt{n}}(1.96)$$

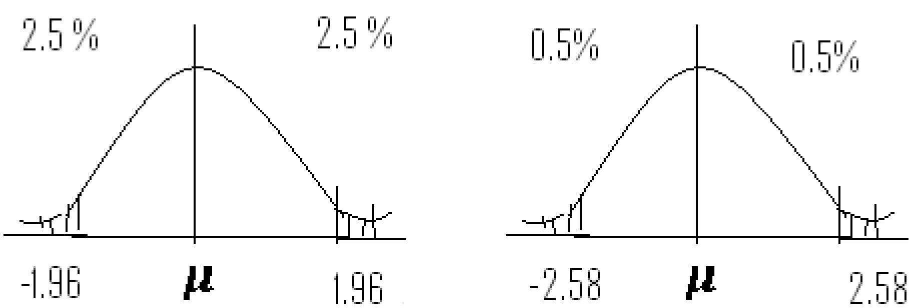
$$\mu \leq \bar{x} + \frac{-\sigma}{\sqrt{n}}(1.96) \;\; and \; \bar{x} - \frac{-\sigma}{\sqrt{n}}(1.96) \leq \mu$$

$\bar{x} - \frac{-\sigma}{\sqrt{n}}(1.96) \leq \mu \;\leq \bar{x} + \frac{-\sigma}{\sqrt{n}}(1.96)$ -----------(2)

Similarly from the table of the normal areas 99% of the area lies between -2.58 and 2.58. This is equivalent to the form,

Therefore representation (2) is that 95% confidence interval and Representation (3) is the 99% confidence level.

Graph:



Tests of significance for large samples:

Let N be the large sample having n members. Let p and q denote number of success and failure respectively, then p+ q = 1. By binomial distribution, $N (p + q)^n$ denotes the frequencies of samples. Therefore $N (p + q)^n$ denotes the sampling distribution of the number of successes in the sample.

We know that by binomial distribution $\bar{x} = np \; and \; \sigma = \sqrt{npq}$ and then, Mean proportion of successes = $\frac{np}{n} = p$

Standard deviation or Standard Error proportion of successes=$\frac{\sqrt{npq}}{n}$

Let ‘x’ be the observed number of successes in a sample size

(n) and μ = np.

The standard normal variate Z is defined as,

$$z = \frac{\bar{x}-\mu}{\sigma} = \frac{\bar{x}-np}{\sqrt{npq}}$$

If Z ≤ 2.58, we conclude that the differences is highly significant

and reject the hypothesis. Then $p \pm 2.58\sqrt{\frac{pq}{n}}$ be the probable limits of z.

Fora normal distribution, only 5% of members lie outside μ ± 1.96 σ while only 1% of the members lie outside μ ± 2.58 σ If x be the observed number of successes in the sample and Z is the standard normal variate

We have the following test of significance

- If Z < 1.96, difference between the observed and expected number of successes is not significant.
- If Z > 1.96 difference is significant at 5% level of significance.
- If Z > 2.58, difference is significant at 1% level of significance.

Examples on Significance of proportion

Example :1

A coin is tossed 1000 times and it turns up head 540 times , decide on the hypothesis is un biased .

Solution:

Let us suppose that the coin is unbiased

P = probability of getting a head in one toss = 1/2

Since p + q = 1,

Expected number of heads in 1000 tosses } = np

= 1000 × 0.5= 500

Actual Number of $heads = 540 = x\ then\ x - np = 540 - 500 = 40$

Consider $z=\frac{x-\mu}{\sqrt{npq}} = \frac{x-np}{\sqrt{npq}} = \frac{540-500}{\sqrt{1000X0.5X0.5}}$

2.53<2.58

$z = 2.53 < 2.58 \rightrightarrows$ 99% (Under)

$\rightrightarrows$ The Coin is unbiased

Example :2

A survey was conducted in one locality of 2000 families by selecting a sample size 800. It was revealed that 180 families were illiterates. Find the probable limits of the literate families in a population of 2000.

Solution: Probability of illiterate families = P = 180/800 = 0.225

Also q = 1 − P = 1 − 0.225 = 0.775

Probability limits of illiterate families = $p \pm 2.58\sqrt{\frac{pq}{n}}$

$= 0.225 \pm 2.58\sqrt{\frac{0.225X0.775}{800}} = 0.187\ and\ 0.263$

Therefore Probable limits of illiterate families in a sample of 2000 is= 0.187(2000) and 0.263(2000)

= 374 and 526

Example:3

A die was thrown 9000 times and a throw of 5 or 6 was obtained 3240 times. On the assumption of random throwing, do the data indicate an unbiased die.

Solution:

Suppose ‘the die is unbiased’

then Probability of throwing 5 or 6 with one die

= p(5) or p(6) = p(5) + p(6) = (1/6) + (1/6) = 1/3 q = 1-p = 1- (1/3) = 2/3

Then expected number of successes (np) = μ (say)

But the observed value of successes = 3240

$0.33 \times 9000 = 3000$

Excess of observed value of successes = $x - np = 3240 - 3000$

= 240

Here n = 9000,
p =1/3 , , np = 3000

Sd=$\sqrt{npq} = \sqrt{9000 * {}^1/_3 * {}^2/_3}$

=44.72

z=$\frac{x-\mu}{\sqrt{npq}} = \frac{240}{44.72} = 5.366 > 2.58$

Highly significant. Hypothesis to be rejected at 1% level of significance . Die is biased.

Example:4

A biased dice is tossed 500 times a particular appears120 times. Find the 95% confidence limit of obtaining the value. Also find the standard error of proportion of success (Use binomial distribution).

Solution:

Let p = $\frac{120}{500}$ = 0.24

then q = 0.76, n = 500.

Standard error = 9.55

Then mean proportion of success = np/n = p = 0.24 and

mean proportion of $\text{S.E} = \sqrt{\frac{npq}{}}$ = 0.019

then 95% confidence interval for proportion of success is

n(0.203) ≤ np ≤ n(0.277)

⇉ 500(0.203) ≤ np ≤ 500(0.277)

101 ≤ np ≤ 138

The interval is [101 , 138].

We say that with 95% confidence that out of 500 times always we get particular number between 101 and 138 times.

Degrees of freedom (d.f)

It is the number of values in a set which may be set arbitrarily.

d.f = n -1 for n number of observations

d.f = n -2 for n -1 number of observations

d.f = n -3 for n - 2 number of observations etc. Ex: for 25 observations we have 24 d.f

5.4 Student's t distribution

It is to test the significance of a sample mean for a normal population where the population S is not known.

Where

$$s^2 = \frac{1}{n-1}\sum(x-\bar{x})^2$$

We need to test the hypothesis, whether the sample mean $\bar{x}$ differs significantly from the population mean μ.

If the calculated value of t i.e. |t| is greater than the table value of t say $t_{0.05}$, we say that the difference between $\bar{x}$ and μ is significant at 5% level.

If |t| > $t_{0.01}$, the difference is significant at 1% level.

Note: 95% confidence limits for the population mean μ.

Example : 5

Ten individuals are chosen at random from a population and their heights in inches are found to be 63, 63, 66, 67, 68, 69, 70, 71,71, test the hypothesis that the mean height of the universe is 66 inches (value of $t_{0.05}$ = 2.262 for 9 d.f).

Solution:

We have μ = 66 , n = 10, ∴ d.f = 9

$$\bar{x} = \frac{\sum x}{n} = \frac{678}{10} = 67.8$$

$$s^2 = \frac{1}{n-1}\sum(x-\bar{x})^2 = 9.067$$

S = 3.011

We have

$$t = \frac{(\bar{x}-\mu)\sqrt{n}}{s} = \frac{(67.8-66)\sqrt{10}}{3.011}$$

$= 1.89 < 2.262$ (given in the problem

$\rightrightarrows$ The hypothesis is accepted at 5% level of significance.

Example: 6

Eleven school boys were given a test in drawing. They were given a month's further tuition and a second test of equal difficulty was healed at the end of it do the marks give evidence that the students have benefitted by extra coaching ($t_{0.05}$ for d.f = 10) = 2.228

Boys	11	2	3	4	5	6	7	8	9	10	11
Marks test 1	23	20	19	21	18	20	18	17	23	16	19
Marks test 2	24	19	22	18	20	22	20	20	23	20	17

Example:6

A population of the four numbers 3,7,11,15. Consider all possible samples of size 2 which can be drawn from this population without replacement. Find the mean and the standard deviation in the population and in the sampling distribution of means.

Population consists of 4 numbers 3,7,11,15.

Mean of the population is $\mu = \frac{3+7+11+15}{4} = 9 - - - - (1)$

Variance of the population is $\sigma^2 = \frac{(3-9)^2+(7-9)^2+(11-9)^2+(15-9)^2}{4} = 20.$

$$\sigma = \sqrt{20} - - - - - (2)$$

Possible samples of size two which can be drqwn without replacement from the given population are

(3,7),(3,11), (3,15),(7,11), (7,15),(11,15).

Means of 6 samples 5,7,9,9,11,13

These are the items in the sampling distribution of means without replacement.

For this mean $\mu_{\bar{x}} = \frac{5+7+9+9+11+13}{6}$=9 -------(3)

Variance =20/3-----(4)

(1) =(3) = 9

$$\frac{\sigma}{\sqrt{N}} \frac{\sqrt{N_p - N}}{\sqrt{N_p - 1}} = \frac{\sqrt{20}}{\sqrt{2}} \frac{\sqrt{4-2}}{\sqrt{4-1}} = \frac{\sqrt{20}}{\sqrt{3}}$$

Example: 7

The daily wages of 3000 workers in a factory are normally distributed with mean equal to Rs.68 and standard deviation equal to Rs 3.If 80 samples consisting of 25 workers each are obtained, what would be the mean and standard deviation of the sampling distribution of means if sampling were done a) with replacement b) without replacement ?

In how many samples will the mean is likely to be i) between Rs 66.8 and Rs 68.3 ii) less than Rs.66.4?

Solution:

$$number\ of\ items\ in\ the\ population\ N_p = 3000$$

Sample size N=25 population mean is $\mu = 68.$

Population standard deviation $\sigma = 3.$

In the case of sampling with replacement, the mean and std. deviation of the sampling deviation of means are given by $\mu = \mu_{\bar{x}} = 68\ \ and\ \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}} = \frac{3}{\sqrt{25}} = 0.6$

If the sampling is done without replacement, the same quantities are given by $\mu = \mu_{\bar{x}} = 68$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}\frac{\sqrt{N_p - N}}{\sqrt{N_p - 1}} = \frac{3}{\sqrt{25}}\frac{\sqrt{3000 - 25}}{\sqrt{3000 - 1}} = 0.5976 = 0.6$$

$\mu_{\bar{x}} = \sigma_{\bar{x}}$=0.6 in both cases.

Since the population is normally distributed , the samplimg distribution of means is also taken as normally distributed.

$$z = \frac{\bar{X} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{\bar{X} - 68}{0.6}$$

For $\bar{X} = 66.8, we\ get\ z = -2\ ; for\ \bar{X} = 68.3, we\ get\ z = 0.5\ and\ for\ \bar{X} = 66.4\ we\ get\ z = -2.67$

The probability that a sample will have a mean between 66.8 and 68.3 is

P(66.8 <$\bar{X} < 68.3) = P(-2 < z < 0.5) = P(0 < z < 2) + P(0 < z < 0.5)$

= A(2) + A(0.5) = 0.4772 + 0.1915 = 0.6687

Accordingly, in 80 samples, the expected number of samples having means between Rs. 66.8 and Rs 68.3 is 0.6687*80 = 53.

Next , the probability that a sample will have a mean less than 66.4 is

P($\bar{X} = 66.4) = P(< -2.67) = P(z > 2.67) = P(0 < z < \infty) - P(0 < z < 2.67)$

=0.5 – A(2.67) =0.5-0.4962 = 0.0038.

Accordingly, in 80 samples, the expected number of samples having means between Rs. 66.8 and Rs 68.3 is 0.0038*80 = 0.304 ≈ 0

Example :8

If the mean of an infinite population is 575 with standard deviation 8.3. How large a sample must be used in order that there be one chance in 100 that the mean of the sample is less than 572?

Solution:

P($\bar{X} < 572) = \frac{1}{100}$

For an infinite population , the standard normal variate associated with $\bar{X}\ is$

$$z = \frac{\bar{X} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{\bar{X} - \mu}{\sigma/\sqrt{N}} = \frac{(\bar{X} - 575)\sqrt{N}}{8.3}$$

$$\bar{X} = 572, we\ get\ z = \frac{-3\sqrt{N}}{8.3} = -0.361\sqrt{N}$$

$\bar{X} < 572\ whenever\ z < -0.361\sqrt{N}.$

P($\bar{X} < 572) = 0.01. we\ should\ have$

P($z < -0.361\sqrt{N}) = 0.01$

P($z > 0) - P(0 < z < 0.361\sqrt{N}) = 0.01$

0.5 – A(0.361$\sqrt{N}$) $= 0.01$

A(0.361$\sqrt{N}$) $= 0.49$

From the normal probability table, we find that A(z)=0.49 when z$\approx$ 2.35

$0.361\sqrt{N} = 2.35\ (or)\ \sqrt{N} = \frac{2.35}{0.361} = 6.51$

N=42.38

Thus , the required sample size must be about 43.

Example :9

Find the probability that in 100 tosses of a fair coin between 45% and 55% of the outcomes are heads.

Solution:

N=100 from the infinite of all possible tosses of the coin.

Since the probability of getting a head in a toss is p = 0.5. the mean and standard deviation for the distribution of the proportion = *π of success in the given sample are*

$$\mu_{\bar{p}} = p = 0.5$$

$$\sigma_p = \left[\frac{p(1-p)}{N}\right]^{1/2} = \left[\frac{0.5(0.5)}{100}\right]^{1/2} = 0.05$$

The corresponding standard normal variate is

$$z = \frac{\bar{X} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{P - 0.5}{0.05}$$

For P=45%, we have $z = \frac{0.45-0.5}{0.05} = -1$

For P=55%, we have $z = \frac{0.55-0.5}{0.05} = 1$

In the chosen sample of tosses, the probability that between 45% and 55% of the outcomes are heads is

*P(0.45<p<0.55)=P(-1<z<1)=2P(0<z<1)=2A(1.00)=2*0.3413=0.6826.*

Example :10

Out of 1000 samples of 200 children each, in how many would you expect to find that a) less than 40% are boys b) between 40% and 60% are boys c) 55% or more are girls.

Solution:

N=200 children from the infinite of all possible tosses of the coin.

Since the probability of getting a boys is p = 0.5. the mean and standard deviation for the distribution of the proportion = π *of success in the given sample are*

$$\mu_{\bar{p}} = p = 0.5$$

$$\sigma_p = \left[\frac{p(1-p)}{N}\right]^{1/2} = \left[\frac{0.5(0.5)}{200}\right]^{1/2} = 0.0354$$

The corresponding standard normal variate is

$$z = \frac{\bar{X} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{P - 0.5}{0.0354}$$

For P=40%, we have $z = \frac{0.4-0.5}{0.0354} = -2.82$

For P=60%, we have $z = \frac{0.6-0.5}{0.0354} = 2.82$

For P=45%, we have $z = \frac{0.45-0.5}{0.0354} = -1.41$

In the chosen sample of tosses, the probability that contains less than 40% of boys is

P(p<0.4)=P(z<-2.82)=P(z>2.82)=P(z>0) – P(0<z<2.82)

=0.5 –A(2.82) =0.5-0.4974 = 0.0026.

*Out of 1000 smples, the expected number of samples containing less htan 40% of boys is 0.0026*1000=2.6*≈ 3

The probability that sample contains between 40% and 60% of boys is

P(0.4< p < 0.6)=P(-2.82 < z < 2.82)=2P(0< z < 2.82)=2A(2.82)

*=2*0.4974=0.9948.*

For 1000 samples =1000X0.9948≈ 995

The prob that a sample contains 55% or more of girls is the same sa the prob of having boys less than 45%.

P(z<0.45)= P(z<-1.41)=P(z>1.41)

=0.5-A(1.41)=0.5-0.4192=0.0808

For 1000 samples =1000X0.0808$\approx 81.$

5.5 Chi-square Distribution:

Suppose a fair coin is tossed 100 times. Then, theoretically speaking, we expect that the coin will show head 50% of times and tail 50%.But this does not happen in practice.In general the coin shows a head in 55 tosses, we say that 55 is the observed frequency of the event of the coin showing a head while the expected frequency ids 50.45 is the observed frequency of a tail while its expected frequency is 50.

In random trails , there exists some discrepancy between the expected frequencies and the observed frequencies.

The discrepancy is analysed through a test statistic called the Chi-square, denoted by χ^2.

Suppose that ,in a random experiment , a set of events $E_1, E_2, E_3, \ldots \ldots E_n$ *are observed to occur with the frequencies* $f_1, f_2, f_3, \ldots \ldots f_n$. According to a theory based on probability rules, suppose the same events are expected to occur with frequencies $e_1, e_2, \ldots\ldots e_n$ are called expected or theoretical frequencies.

$$\chi^2 = \frac{(f_1-e_1)^2}{e_1} + \frac{(f_2-e_2)^2}{e_2} + \cdots \ldots \ldots + \frac{(f_n-e_n)^2}{e_n} = \sum_{k=1}^{n} \frac{(f_k-e_k)^2}{e_k} \text{-------------(1).}$$

If N is the total frequency, we should have

$$N = \sum_{k=1}^{n} f_{k=} \sum_{k=1}^{n} e_k \; --- \; (2)$$

If the expected frequencies are atleast equal to 5, then it can be proved that the sampling distribution fo the statistics χ^2 *whose density function is given by*

$$P(\chi^2) = P_0 \chi^{v-2} e^{\left(-\chi^2/2\right)} \; ------ (3).$$

Where v is a positive constant called the number of degree of freedom and P_0 *is a constant depending on v such that the total area under the corresponding probability curve is one.*

The probability distribution for which given by (3) is the density function is called the **Chi-Square distribution** with v degrees of freedom.

Chi-Square Test

In practice, expected frequencies are computed on the basis of hypothesis H_0. If under this hypothesis the value of Using the formula

χ^2 *computed with the use of the formula* (1) *is grater than the critical value* χ_c^2
.
We would conclude that the observed frequencies differ significantly from the expected frequencies and would reject H_0 at the corresponding level of significance c. otherwise we would accept it or atleast not reject it.
This procedure is called the Chi-square Test of hypothesis or significance.

Generally, the Chi-Square test is considered by taking critical value c=0.05 or 0.01.
Goodness of Fit

When a hypothesis H_0 is accepted on the basis of the chi-square test then the expected frequencies calculated on the basis of H_0 form a good fit for the given frequencies.

Chi-Square distribution: χ^2

It provides a measure of correspondence between the Theoretical frequencies and observed frequencies
Let O_i (i = 1 , 2 , ….. n) – observed frequencies $\boldsymbol{e_i}$ (i = 1 , 2 , ….. n) – estimated frequencies

The quantity χ^2 (chi square) distribution is defined as

; degrees of freedom = n-1

$$\chi^2 = \frac{\sum_{i=1}^{n}(o_i - e_i)^2}{e_i}$$

Chi – square test as a test of goodness of fit:

χ^2 test helps us to test the goodness of fit of the distributions such as Binomial, Poisson and Normal distributions.

If the calculated value of χ^2 is less than the table value of χ^2 at a specified level of significance, the hypothesis is accepted. Otherwise the hypothesis is rejected.

Example :11
If 200 tosses of a coin, 118 heads and 82 tails were observed. Test the hypothesis that the coin is fair at 0.05 and 0.01 levels of significance.
Solution:

Observed frequencies of head f_1=118 and tails f_2 = 82respectively.

N=200=number of trails.

Expected frequencies of heads e_1=200X0.5=100

Expected frequencies of tails e_2=200X0.5=100

Sum of expected frequencies = Sum of observed frequencies

$$\chi^2 = \frac{(f_1 - e_1)^2}{e_1} + \frac{(f_2 - e_2)^2}{e_2} = \frac{(118 - 100)^2}{e_1} + \frac{(82 - 100)^2}{100} = 6.48$$

N=200 is the only quantity used under this computation of e_i.

Number of degree of freedom v = n-1=1 (where n=2 frequency pairs are used)

$$\chi^2_{0.05}(1) = 3.84\ , \chi^2_{0.01}(1) = 6.64$$

$$\chi^2 > \chi^2_{0.05}(1) = 6.64\ , \chi^2 < \chi^2_{0.01}(1) = 3.84$$

Using the Chi-square Test, we accept the hypothesis that the coin is fair at 0.01 level of significance, but do not accept it at 0.05 level of significance.

Example :12

A die is thrown 60 times and the frequency distribution for the number appearing on the face x is given by

X	1	2	3	4	5	6
frequency	15	6	4	7	11	17

Test the hypothesis that the die is unbiased.

Solution:

If the die is unbiased, hten every number has equal probability of appearing the face x.

Expected frequency in N = 60 throws is 60X(1/6)=10.

$e_1 = e_2 = e_3 = e_4 = e_5 = e_6 = 10$

The observed frequencies are f_1 = 15, f_2=6 f_3 =4, f_4 =7 f_5 =11, f_6 =17.

N=60 = Sum of expected frequencies = Sum of observed frequencies

$$\sum_{k=1}^{6}\frac{(f_k-e_k)^2}{e_k}=\frac{1}{10}[(15-10)^2+(6-10)^2+(4-10)^2+(7-10)^2 + (11-10)^2+(17-10)^2]=13.6$$

Note n = 6 frequency pairs are used in computation of χ^2, *and* $N = 60$ *is the only quantity used in the computation of* e_i

Number of degrees of freedom =v =6-1=5.

$$\chi^2_{0.05}(5)=11.04\,,\chi^2_{0.01}(5)=15.09$$

$$\chi^2=13.6>\chi^2_{0.05}(5)=15.09\,,\chi^2=13.6<\ \chi^2_{0.01}(5)=11.07$$

Using the Chi-square Test, we accept the hypothesis that the die is unbiased is fair at 0.01 level of significance, but do not accept it at 0.05 level of significance.

Example :13

A set of five identical coins is tossed 320 times and the results is shown in the table

No. of heads	0	1	2	3	4	5
Frequency	6	27	72	112	71	32

Test the hypothesis that the data follows a binomial distribution associated with a fair coin.

P(x) = $\binom{5}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{5-x}=\frac{1}{2^5}\binom{5}{x}=b(x)$

In 320 tosses the expected number of tosses in which x number of coins show a head is 320Xb(x).

Ie e_1=320Xb(0) = 320 $\frac{1}{2^5}\binom{5}{0}=10$ e_2 = 320Xb(1) = 320 $\frac{1}{2^5}\binom{5}{1}=50$, e_3 = 320Xb(2) = 320 $\frac{1}{2^5}\binom{5}{2}=100$

e_4 = 320Xb(3) = 320 $\frac{1}{2^5}\binom{5}{3}=100$, e_5 = 320Xb(4) = 320 $\frac{1}{2^5}\binom{5}{4}=50$, e_6 = 320Xb(5) = 320 $\frac{1}{2^5}\binom{5}{5}=10$

$$\chi^2=\frac{(6-10)^2}{10}+\frac{(27-50)^2}{50}+\frac{(72-100)^2}{100}+\frac{(112-100)^2}{100}+\frac{(71-50)^2}{50}+\frac{(32-10)^2}{50}=78.68$$

Note n = 6 frequency pairs are used in computation of

χ^2, *and* $N = 320$ *is the only quantity used in the computation of* e_i

Number of degrees of freedom =v =6-1=5.

$$\chi^2_{0.05}(5) = 11.04\,,\chi^2_{0.01}(5) = 15.09$$

$$\chi^2 = 78.68 > \chi^2_{0.05}(5) = 11.04\,,\chi^2_{0.01}(5) = 15.09$$

We reject the hypothesis that the observed data follows a binomial distribution associated with a fair coin.

Example :14

A set of five identical coins is tossed 100 times and the results is shown in the table

No. of heads	0	1	2	3	4	5
Frequency	2	14	20	34	22	8

Test the goodness of this fit at 5% level of significance.

Solution:

$$\sum xf(x) = 284$$

$$Mean = \frac{\sum xf(x)}{\sum f} = \frac{284}{100} = 2.84$$

Mean of Binomial distribution = np

2.84= np=6p

P=0.568

P(x) = $\binom{5}{x}(0.568)^x(0.432)^{5-x} = b(x)$

In 100 tosses the expected number of tosses in which x number of coins show a head is 100Xb(x).

Ie e_1=100Xb(0) = 100 $\binom{5}{0}(0.568)^0(0.432)^{5-0} = 1.505$

e_2 = 1000X$\binom{5}{1}(0.568)^1(0.432)^{5-1}$) = 10.4512

e_2 = 100Xb(2) = 100 $\binom{5}{2}(0.568)^2(0.432)^{5-2} = 26.01$

e_3=100Xb(3) =100 $\binom{5}{3}(0.568)^3(0.432)^{5-3} = 34.199$

e_4 = 100Xb(4) = 100 $\binom{5}{4}(0.568)^4(0.432)^{5-4} = 22.483$,

e_5 = 100Xb(5) = 100 $\binom{5}{5}(0.568)^5(0.432)^{5-5} = 5.912$

m=2   mean of frequency and the sum of frequency have been used.

Sum of observed frequency=100
Sum of theoretical frequencies =100.5602 = $\sum xe$

e1 is very less hence add e1 and e2 ie e1+e2 = 11.9562
f1+f2 = 16
e_3 is large difference between observed and theoretical frequencies

$$\chi^2 = \frac{(16-11.9562)^2}{11.9562} + \frac{(20-25.4498)^2}{25.4498} + \frac{(34-34.199)^2}{34.199} + \frac{(22-22.483)^2}{22.483} + \frac{(8-5.912)^2}{5.912} = 3.28$$

Note n = 5 frequency pairs are used in computation of $\chi^2, and\ N = 100$
Number of degrees of freedom =v =5-2= 3.

$$\chi^2_{0.05}(3) = 7.82$$

$$\chi^2 = 3.28 < \chi^2_{0.05}(3) = 7.82$$

$$We\ accept\ the\ goodness\ of\ fit\ at\ 5\%\ a\ binomial\ distribution\ .$$

Example:15

A die is thrown 264 times and the number appearing on the face (x) follows the following frequency distribution

x	1	2	3	4	5	6
f	40	32	28	58	54	60

Calculate the value of χ^2

Solution:

Frequencies in the given table are the observed frequencies.

Assuming that the die is unbiased the expected number of frequencies for the numbers 1, 2, 3,4,5,6 to appear on the face is 264/6 = 44 each

Then the data is as follows

No. on the die	1	2	3	4	5	6
Observed frequency(O_i)	40	32	28	58	54	60
Expected frequency(E_i)	44	44	44	44	44	44

$$\chi^2 = \frac{\sum_{i=1}^{n}(o_i - e_i)^2}{e_i}$$

$$= \frac{(40-44)^2}{44} + \frac{(32-44)^2}{44} + \frac{(28-44)^2}{44} + \frac{(58-44)^2}{44} \frac{(54-44)^2}{44} + \frac{(60-44)^2}{44}$$

=22

Example:16

Fit the Poisson distribution for the following data and test the goodness of fit given that $\chi^2_{0.05}$= 7.815 for degrees of freedom = 4

x	0	1	2	3	4
f	122	60	15	2	1

Solution:

$\frac{\sum fx}{\sum f} = \mu = \frac{0+60+30+6+4}{200} = 0.5$

$$p(x) = m^x \frac{e^{-m}}{x!}$$

Let f(x) = 200.p(x)

$$f(x) = 200 \frac{(0.5)^x e^{-0.5}}{x!}$$

$$f(x) = 121.3 \frac{(0.5)^x}{x!}$$

X = 0,1,2,3,4, in f(x). We obtain the theoretical frequencies. We get

Therefore new table is

x	0	1	2	3	4
$f(o_i)$	120	60	15	2	1
E_i	121	61	15	3	0

$\chi^2 \quad = 0.025 < 3^2{}_{0.05} = 7.815$

Therefore the fitness is considered good.

∴ The hypothesis that the fitness is good can be accepted.

Example:17

In experiments of pea breeding, the following frequencies of seeds were obtained

Round & yellow	Wrinkled & yellow	Round & green	Wrinkled & green	total
315	101	108	32	556

Theory predicts that the frequency should be in proportion 9:3:3:1. Examine the correspondence between theory and experiment.

Solution:

Corresponding frequencies are 313, 104, 104, 35.

$\chi^2 \quad = 0.51 < 3^2{}_{0.05} = 7.815$

⟹ The calculated value of 3^2 is much less than $3^2{}_{0.05}$

⟹ There exists agreement between theory and experiment.

We considered sampling distribution on the assumption that they are normal. When sample size N is large. For small samples , this assumption is not generally valid.
T- distrintuion which is used in small samples.

Let N =small size, $\bar{x}$ and μbe sample mean and population mean.
S be the sample standard deviation.

$$t = \frac{\bar{x} - \mu}{s}\sqrt{N-1} - - - - - - (1)$$

Students t distribution is known as a frequency distribution of t by computing the value of t for each of a set of samples of size N drawn from a normal

Y(t)=y$_0$$\left(1 + \frac{t^2}{N-1}\right)^{\frac{-N}{2}}$

Y_0 is an appropriate constant is called t-curve. Y_0 is generally chosen is such a way that the total area under the curve is equal to unity.

For large value of N y(t) reduces to standard normal distribution t curve become normal curve.

Confidence limits are given by $\bar{x} \pm t_c\left(\frac{s}{\sqrt{N-1}}\right)$

Where $\pm t_c$ are the critical values or confidence coefficients whose values depend on level of significance desired and the sample size.

Example:18

For a random sample of 16 values with mean 41.5 and the sum of squares of the deviations from the mean equal to 135 and drawn from a normal distribution. Find the 95% confidence limits and the coefficient interval, for the mean of the population.

Solution:

N=16 so degree of freedom N-1=15

95% confidence level =$t_{0.05}(15) = 2.13$
Sample mean = 41.5

$$s^2 = \frac{1}{16}(135) = 8.4375$$

The required confidence limits are $\bar{x} \pm t_c\left(\frac{s}{\sqrt{N-1}}\right) = 41.5 \pm \left(\frac{\sqrt{8.4375}}{\sqrt{15}} X 2.13\right)$

=41.5$\pm$1.5975 = 43.1,39.9
95% confidence level is (39.9,43.1).